Please provide complete and well-written solutions to the following exercises.

No due date, but the quiz in Week 2 in the discussion section (on August 30) will be based upon this homework.

Q2: Quiz 2 Problems

Exercise 1. Define

$$H(x) = \begin{cases} 0 & \text{, if } x < 0 \\ 1 & \text{, if } x \geq 0 \end{cases}.$$

Explain in your own words why $\lim_{x\to 0} H(x)$ does not exist.

Exercise 2. Find two functions f, g such that $\lim_{x\to a} f(x)$ does not exist, $\lim_{x\to a} g(x)$ does not exist, but such that

$$\lim_{x\to a}(f(x) + g(x))$$

does exist.

Exercise 3. Evaluate the following limit and justify each step by indicating the appropriate limit law.

$$\lim_{u\to -2} \sqrt{u^4 + 3u + 6}$$

Exercise 4. Evaluate the following limit, if it exists. If it does not exist, explain why it does not exist.

$$\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right)$$

Exercise 5. Evaluate the following limit, if it exists. If it does not exist, explain why it does not exist.

$$\lim_{x\to 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

Exercise 6. Is there a real number a such that the following limit exists?

$$\lim_{x\to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

If so, find the value of a and the value of the limit.

Exercise 7. Are the following statements true or false?

(a) If $\lim_{x\to 5} f(x) = 0$ and $\lim_{x\to 5} g(x) = 0$, then $\lim_{x\to 5} \frac{f(x)}{g(x)}$ does not exist.
(b) If x is a real number, then $\sqrt{x^2} = x$
(c) If $\lim_{x\to 5} f(x) = 2$ and $\lim_{x\to 5} g(x) = 0$, then $\lim_{x\to 5} \frac{f(x)}{g(x)}$ does not exist.
(d) If f is continuous at 5 and $f(5) = 2$, then $\lim_{x\to 2} f(4x^2 - 11) = 2$.
(e) If $f(x) > 1$ for all $x \neq 0$ and $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0} f(x) > 1$.

Exercise 8. Fix $x \in \mathbf{R}$, and let $f(x) = x^2$. Calculate the following limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The fraction $(f(x+h)-f(x))/h$ is known as a difference quotient. The limit of this difference quotient will come up again later in the course.

Exercise 9. Let $f, g \colon \mathbf{R} \to \mathbf{R}$ and let $a \in \mathbf{R}$. Is it always true that $\lim_{x \to a}(f(x) + g(x)) = (\lim_{x \to a} f(x)) + (\lim_{x \to a} g(x))$?

Exercise 10. Find all values of a and b such that the following function is continuous:

$$f(x) = \begin{cases} ax - b & x \leq -1 \\ 2x^2 + 3ax + b & -1 < x \leq 1 \\ 4 & x > 1 \end{cases}.$$

Exercise 11. For what values of x is the following function continuous: $g(x) = (3x^5 + 10)^{1/3}$. (Hint: treat each function as a composite function, and look at the domain of each part.)

Exercise 12. Draw the following set and describe it in words: the set of all points (x, y) in the plane such that

$$\lim_{t \to \infty} (|x|^t + |y|^t) < 4.$$