Please provide complete and well-written solutions to the following exercises.

No due date, but the quiz in Week 13 in the discussion section (on November 15) will be based upon this homework.

Q11: Quiz 11 Problems

Exercise 1. Find the maximum and minimum values of the function $f(x,y)=x^2+yx$ on the disk $x^2+y^2\leq 1$.

Exercise 2. In statistics and other applications, we can be presented with data points $(x_1,y_1),\ldots,(x_n,y_n)$. We would like to find the line $y=mx+b$ which lies "closest" to all of these data points. Such a line is known as a **linear regression**. There are many ways to define the "closest" such line. The standard method is to use **least squares minimization**. A line which lies close to all of the data points should make the quantities (y_i-mx_i-b) all very small. We would like to find numbers m,b such that the following quantity is minimized:

$$f(m,b)=\sum_{i=1}^{n}(y_i-mx_i-b)^2.$$

Using the second derivative test, show that the minimum value of f is achieved when

$$m=\frac{\left(\sum_{i=1}^{n}x_i\right)\left(\sum_{j=1}^{n}y_j\right)-n\left(\sum_{k=1}^{n}x_ky_k\right)}{\left(\sum_{i=1}^{n}x_i\right)^2-n\left(\sum_{j=1}^{n}x_j^2\right)}.$$

$$b=\frac{1}{n}\left(\sum_{i=1}^{n}y_i-m\sum_{j=1}^{n}x_j\right).$$

Briefly explain why this is actually the minimum value of $f(m,b)$. (You are allowed to use the inequality $(\sum_{i=1}^{n}x_i)^2\leq n(\sum_{i=1}^{n}x_i^2)$.)

Exercise 3. Let $f(x,y)=x^2+y^2$. Using Lagrange Multipliers, find the maximum and minimum of f on the ellipse $x^2+2y^2=1$.

Exercise 4. Let $f(x,y)=x^2+y$. Using Lagrange Multipliers, find the maximum and minimum of f on the circle $x^2+y^2=1$.

Exercise 5. Let a,b,c be positive constants. An ice cream cone is defined as the surface $z=a\sqrt{x^2+y^2}$ where $z\leq b$. Suppose the ice cream cone has surface area c. Using Lagrange Multipliers, find the ice cream cone of fixed surface area c and with maximum volume. (This way, you get to eat the most ice cream with the least amount of material.) You should probably use the variables r and b, where r denotes the radius of the cone, and where b is the height of the cone. You can freely use that the surface area of the cone is $\pi r\sqrt{r^2+b^2}$.