Please provide complete and well-written solutions to the following exercises.

No due date, but the quiz in Week 12 in the discussion section (on November 8) will be based upon this homework.

Q10: Quiz 10 Problems

Exercise 1. Let $f(x,y) = x^2 + y^2$. Compute the partial derivatives: f_{xx}, f_{xy}, f_{yx}, f_{yy}.

Exercise 2. Let $f(u,v,w,x,y,z) = u^2/v + vxyz + e^{xwv}$. Compute the partial derivatives: f_{uv}, f_{wz}, f_{xyz}.

Exercise 3. Consider the following function $f\colon \mathbf{R}^2 \to \mathbf{R}$.

$$f(x,t) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}, \qquad t > 0.$$

Show that f satisfies the **heat equation** (for one spatial dimension x):

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}.$$

The function f represents a single point of heat emanating through an infinite rod (the x-axis) as time passes (as t increases, $t \geq 0$). The heat equation roughly says that the rate of change of heat f at the point x and at time t is equal to the average difference between the current heat at x, and the neighbors of x. The quantity $\partial f/\partial t$ is the rate of change of heat, while the second derivative on the right is perhaps better understood using the second-difference quotient:

$$\partial^2 f/\partial x^2 = \lim_{h\to 0} \frac{f(x-h,t) - 2f(x,t) + f(x+h,t)}{h^2}.$$

Exercise 4. Let $f(x,y,z) = x^2 + y^3$. Compute the gradient $\nabla f(x,y)$. Find the linearization of f at the point $(a,b) = (1,2)$. Using this linearization and the approximation $f(x,y) \approx L(x,y)$, approximate the quantity $f(1.1, 1.9)$.

Exercise 5. Let $f(x,y) = x^2 y^3$. Compute the gradient $\nabla f(x,y)$. Then, find the tangent plane to the surface $z = f(x,y)$ at the point $(a,b) = (1,2)$.

Exercise 6. Let $f(x,y) = x^2 + y^2$ and let $g(x,y) = x^2 - y^2$. For any point (x,y) in the plane, we can plot the vector $\nabla f(x,y)$ in the plane, so that $\nabla f(x,y)$ has basepoint (x,y). Plotting the vector $\nabla f(x,y)$ in this way for many values of (x,y) allows us to visualize the gradient $\nabla f(x,y)$. For any x in the set $-2,-1,0,1,2$, and for any y in the set $-2,-1,0,1,2$, plot the gradient vector $\nabla f(x,y)$. Then, in a separate drawing, plot the gradient vector $\nabla g(x,y)$.

Exercise 7. It is the zombie apocalypse. It is safer at the moment to run to higher ground. The height of the land nearby is proportional to the function $f(x,y) = e^{-(x^2+y^2)/2} + xy^3$. You are located at the point $(x,y) = (1,-1)$. In which direction should you run if you want to immediately:

- Move to higher ground?
- Stay at the same elevation?
- Move to lower ground?

Exercise 8. Let $f(x, y) = x^2 + xy + y$. Identify the critical points of f, and identify these points as local maxima, local minima or saddle points.

Exercise 9. Find the maximum and minimum values of the function $f(x, y) = e^{-x^2-y^2}$ in the plane, if they exist. If they do not exist, briefly explain why.