118 Fall 2017 Final Solutions[1]

1. Question 1

The growth of the Starbucks stores over the years is shown in the figure below. The number of stores in the US and outside of the US are represented by the heights of the white and shaded bars, respectively

(a) Find the average rate of growth m of the chain worldwide (including US) between 2010 and 2017.

Solution. In 2010, the number of stores is $11131 + 5727$ and in 2017, the number of stores is $13930 + 13409$. So, the average rate of growth between these years is

$$\frac{13930 + 13409 - (11131 + 5727)}{2017 - 2010} = \frac{27339 - 16858}{7} = \frac{10481}{7} \approx 1497 \text{ stores per year}$$

(b) Assuming the above rate of growth m is maintained in the future, what would be the number of Starbucks stores worldwide by 2020?

Solution. The number of stores in 2017 is 27339. Assuming a rate of growth of 1497 stores per year, in 2020 there would be

$$27339 + (2020 - 2017)(1497) = 27339 + (3)(1497) = 31830 \text{ stores}$$

(c) If the growth from 2010 is assumed to be exponential, use the data from 2010 and 2017 to find the exponential rate of growth k, and a formula for the number of chains worldwide t years after 2010.

Solution. If the growth is exponential, then $m(t) = ae^{kt}$ where t is years after 2010 and a, k are constants. So, $m(0) = 16858$ and $m(7) = 27339$. That is, $a = 16858$, so $16585e^{7k} = m(7) = 27339$, so $e^{7k} = 27339/16585$, $7k = \ln(27339/16585)$, and $k = (1/7)\ln(27339/16585)$. In summary,

$$m(t) = 16858e^{\frac{t}{7}\ln(27339/16585)}.$$

(d) If the growth from 2010 is assumed to be exponential, what would be the number of Starbucks stores worldwide by 2020?

Solution. Using part (c), the number of stores in 2020 is $m(10)$, so that

$$m(10) = 16858e^{\frac{10}{7}\ln(27339/16585)} \approx 33634.$$

2. Question 2

Shown below is the graph of the derivative f' of a function f.

(a) For each quantity below, determine if it is positive, negative, equal to 0, or whether the information is insufficient to determine this. Give a brief explanation in each case.

- $f(-1)$. Insufficient information. Any function of the form $f + C$ also satisfies $(f + C)' = f'$, so knowing only the derivative of f means it is impossible to know the value of f at a particular point.
- $f'(-1)$. Positive. $f'(-1)$ is above the x-axis in the picture, so that $f'(-1) > 0$.
- $f'(-1)$. Positive. f' has positive slope in the picture at -1, so that $f''(-1) > 0$.

(b) Consider the interval $(1, 0)$. Determine if each function below is increasing or decreasing on this interval. Give a brief explanation in each case.

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- f. Increasing. f' is above the x-axis on $(-1, 0)$ so that f is increasing on $(-1, 0)$.
- f'. Increasing. f' has positive slope in the picture on $(-1, 0)$, so that f' is increasing on $(-1, 0)$
- f'. Decreasing. The slope of f' decreases as x goes from -1 to 0, so that f'' is decreasing on $(-1, 0)$.

(c) For each input value below, choose one of the sketches to show what the graph of the function f looks like in a neighborhood of that point. A sketch may be used once, more than once, or not at all.

- $x = -2$. IV. (f goes down, then it goes up.)
- $x = -1$. II. (The slope of f is increasing.)
- $x = 0$. VII. (The slope of f increases and then decreases.)

3. Question 3

It is estimated that the cost of constructing an office building that is n floors high is

$$C = 3n^2 + 500n + 1810,$$

where C is measured in thousands of dollars. How many floors should a building have in order to minimize the average cost per floor?

(Hint: If the critical point is not an integer, check neighboring integer points.)

Solution. The average cost for floor is $f(n) = C(n)/n = 3n + 500 + 1810/n$. We have $f'(n) = 3 - 1810/n^2$. So, if $f'(n) = 0$, we have $3 - 1810/n^2 = 0$ so that $3n^2 = 1810$ and $|n| = \sqrt{1810/3}$. We only consider $n \geq 0$ since $n < 0$ is unrealistic in this problem. When $0 \leq n < \sqrt{1810/3}$, we have $f'(n) < 0$ and when $n > \sqrt{1810/3}$ we have $f'(n) > 0$. So, f decreases and then increases, so that $\sqrt{1810/3}$ is the global minimum of f when $n \geq 0$. As suggested in the hint, $\sqrt{1810/3}$ is not an integer, so to get a meaningful answer, we have to round n to an integer. It is not clear which nearby integer has a smaller value so we check both. We have $\sqrt{1810/3} \approx 24.5628$, so we check the values 24 and 25. Then $f(24) = 72 + 500 + 1810/24 \approx 647.8333$ and $f(25) = 75 + 500 + 1810/25 \approx 647.8$. So, the average cost is minimized when $n = 25$ floors are constructed.

4. Question 4

Consider this integral of $f(x) = 2x$ with unknown endpoints:

$$F(a, b) = \int_a^b 2x dx.$$

(a) For which values of a and b will $F(a, b) = 0$? Circle all answers that are correct, and leave blank any that are incorrect.

- $a = 2, b = 3$.
- $a = 0, b = 0$. [Circle]
- $a = 50, b = 50$. [Circle]
- $a = 0, b = 1$.
- $a = 3, b = 3$. [Circle]

(b) For which values of a and b will $F(a, b) > 0$? Circle all answers that are correct, and leave blank any that are incorrect.

- $a = 2, b = 3$. [Circle]
- $a = 0, b = 0$.
- $a = 1, b = 2$. [Circle]
- $a = 0, b = 1$. [Circle]
- $a = -2, b = 1$.
- $a = 3, b = 3$.

(c) If we assume that a and b are both positive ($0 < a < b$), then which combination of horizontal width (w) and vertical height (h) describes a rectangle whose area is equal to the area represented by $F(a,b)$? Circle all answers that are correct, and leave blank any that are incorrect.

$F(a,b) = \int_a^b 2x dx = [x^2]_{x=a}^{x=b} = b^2 - a^2$.

- $w = b - a$, $h = b^2 - a^2$. ($wh \neq b^2 - a^2$.)
- $w = b - a$, $h =$ the average value of $f(x) = 2x$ on $[a,b]$. [Circle] $h = \frac{1}{b-a} F(a,b) = \frac{b^2-a^2}{b-a}$, so $wh = b^2 - a^2 = F(a,b)$.
- $w = a$, $h = f(b)$.
- $w = a$, $h =$ the maximum value of $f(x) = 2x$ on $[a,b]$.

5. Question 5

A woman lives on Venice Blvd, 1.2 miles from the ocean. (Venice Blvd runs east-west from downtown LA all the way to the ocean.)

She takes a walk, heading west from her house toward the ocean. The graph below shows her velocity $v(t)$ during her walk, t minutes from when she leaves her house.

(a) From 30 to 40 minutes into her walk, is she walking towards the ocean, or towards her home? Explain.

Solution. She is walking towards the ocean. It is stated in the problem that she begins her walk headed towards the ocean. The picture shows her velocity starting as positive and it stays positive for the first 40 minutes of her walk. So, during the first 40 minutes of her walk, she is headed towards the ocean.

(b) From 40 to 50 minutes into her walk, is she walking towards the ocean, or towards her home? Explain.

Solution. She is walking towards her home. During the first 40 minutes of her walk, she is headed towards the ocean, as noted in part (a). But from minute 40 to minute 50, her velocity is negative, i.e. she reversed her original direction. That is, she is headed towards her home during this time.

(c) Between 0 and 20 minutes into the womans walk, is she speeding up or slowing down? Explain.

Solution. She is speeding up. The picture shows her velocity increasing during this time period.

(d) How many miles is the woman from her home, one hour into her walk?

Solution. Let $s(t)$ be her position at time t, so that $s(t) > 0$ denotes she is west of her house, and $s(t) < 0$ denotes she is east of her house. We let t be minutes. From the Fundamental Theorem of Calculus,

$$s(60) - s(0) = \int_0^{60} s'(t)dt = \int_0^{60} v(t)dt.$$

In order for this formula to be correct, the units of v also have to be in miles per minute, not miles per hour. To do this unit conversion we divide the velocity by 60. From the picture, we conclude that

$$\begin{aligned}\int_0^{60} v(t)dt &= \frac{1}{60}[10(1)(1/2)+10+10(2)(1/2)+10(3)+10(3)(1/2)-10(2)(1/2)-10(2)(1/2)]\\ &= \frac{1}{60}[5+10+10+30+15-10-10] = \frac{1}{60}[70-20] = \frac{50}{60} = 5/6.\end{aligned}$$

So, she is 5/6 miles west of her house after one hour.

6. Question 6

Compute the following integrals.

(a) $\int (7x-1)e^{2x}dx$.

Solution. We integrate by parts using $\int udv = uv - \int vdu$ where $u = 7x-1$ and $v = (1/2)e^{2x}$. Then $dv = e^{2x}dx$ and $du = 7dx$, so

$$\begin{aligned}\int (7x-1)e^{2x}dx = \int udv = uv - \int vdu &= (7x-1)(1/2)e^{2x} - \int (1/2)e^{2x}7dx\\ &= (7x-1)(1/2)e^{2x} - (7/4)e^{2x} + C.\end{aligned}$$

(b) $\int x^2(x^3+1)^4dx$.

Solution. We use u-substitution of the form $u = x^3+1$ so that $du = 3x^2dx$ and

$$\int x^2(x^3+1)^4dx = (1/3)\int u^4du = (1/3)(1/5)u^5 + C = \frac{1}{15}(x^3+1)^5 + C.$$

(c) $\int x^3e^{-x^2/2}dx$.

Solution. We integrate by parts using $\int udv = uv - \int vdu$ where $u = x^2$ and $v = e^{-x^2/2}$. Then $dv = -xe^{-x^2/2}dx$ and $du = 2xdx$, so

$$\int x^3e^{-x^2/2}dx = -\int udv = -uv + \int vdu = -x^2e^{-x^2/2} + \int 2xe^{-x^2/2}dx.$$

We now use a substitution, with $w = x^2/2$, so that $dw = xdx$ and

$$\begin{aligned}\int x^3e^{-x^2/2}dx &= -x^2e^{-x^2/2} + 2\int e^{-w}dw = -x^2e^{-x^2/2} - 2e^{-w} + C\\ &= -x^2e^{-x^2/2} - 2e^{-x^2} + C = -(2+x^2)e^{-x^2/2} + C.\end{aligned}$$

7. Question 7

For each function below, find the indicated partial derivative:

(a) $f(x,y) = \sqrt{1+xy}$.

From the Chain rule,

$$f_x = (1/2)(1+xy)^{-1/2}\frac{\partial}{\partial x}(1+xy) = (1/2)(1+xy)^{-1/2}y = \frac{y}{2\sqrt{1+xy}}.$$

(b) $P(u,v) = u10^{uv}$.

From the Chain rule and $(d/dx)b^x = b^x \ln b$,

$$\frac{\partial P}{\partial v} = u10^{uv}\ln(10)\frac{\partial}{\partial v}(uv) = u10^{uv}\ln(10)u = u^2 10^{uv}\ln(10).$$

(c) $T(x,y) = \frac{x+y}{x-y}$.

From the quotient rule,

$$T_x = \frac{(x-y)\frac{\partial}{\partial x}(x+y) - (x+y)\frac{\partial}{\partial x}(x-y)}{(x-y)^2} = \frac{(x-y)-(x+y)}{(x-y)^2} = \frac{-2y}{(x-y)^2}.$$

From the quotient rule again,

$$T_{xy} = \frac{\partial}{\partial y}T_x = \frac{(x-y)^2\frac{\partial}{\partial y}(-2y) + 2y\frac{\partial}{\partial y}(x-y)^2}{(x-y)^4} = \frac{(x-y)^2(-2) + 2y2(x-y)(-1)}{(x-y)^4}$$

$$= \frac{(x-y)(-2) + 2y2(-1)}{(x-y)^3} = \frac{-2x+2y-4y}{(x-y)^3} = \frac{-2x-2y}{(x-y)^3}.$$

8. Question 8

When a company sells its product for p dollars and spends A dollars per month on advertising, its monthly revenue is $R(p,A)$ dollars. Suppose the company presently sells its product for \$200 and spends \$10,000 per month on advertising.

(a) Also suppose that $\frac{\partial R}{\partial A}|_{(200,10000)} = 1.75$. What does this tell us about the company? Circle only the best answer.

- If the company increases both the selling price of its product and the amount it spends on advertising, then its revenue will increase by \$1.75.
- The companys revenue is presently approximately \$200.
- If the company increases the amount it spends on advertising by \$200, its revenue will increase by approximately \$1.75.
- If the company increases the amount it spends on advertising by \$1, its revenue will increase by approximately \$1.75. [Circle]
- If the company increases its revenue by \$1, then the amount it spends on advertising will increase by approximately \$1.75.

(b) Further suppose that $R(200, 10000) = 125,000$ and $\frac{\partial R}{\partial p}|_{(200,10000)} = 210$.

Estimate the companys monthly revenue if it increases the selling price of the product by \$1.70 and decreases the amount spent on advertising by \$150.

Solution. Using linear approximation, we have

$$R(200+a, 10000+b) \approx R(200,10000) + a\frac{\partial R}{\partial p}|_{(200,10000)} + b\frac{\partial R}{\partial A}|_{(200,10000)}.$$

That is,

$$R(200+1.7, 10000+150) \approx 125,000 + 1.7(-210) + 150(1.75) = 125,00 - 357 + 262.5 = 12405.5.$$

9. Question 9

A company produces two versions of its product – one for domestic markets and one for export. In the domestic market, the demand curve is

$$p_1 = 45 - q_1.$$

while in the export market the demand curve is

$$p_2 = 55 - q_2.$$

[q_1 is the price of the domestic product, q_2 is the price of the export product.]

(a) Write an expression for the company's revenue R in terms of q_1 and q_2.

Solution. $R = p_1q_1 + p_2q_2 = (45 - q_1)q_1 + (55 - q_2)q_2$.

(b) The cost of producing q_1 units of the domestic version and q_2 of the export version is

$$C(q_1, q_2) = 1000 + 5q_1 + 5q_2 + q_1q_2.$$

Find the critical points (q_1, q_2) for the companys profit P, also considered as a function of q_1 and q_2.

Solution. $P = R - C = (45 - q_1)q_1 + (55 - q_2)q_2 - [1000 + 5q_1 + 5q_2 + q_1q_2]$. The critical points must occur when both partial derivatives are zero. That is, we solve

$$\frac{\partial}{\partial q_1}P = 0, \qquad \frac{\partial}{\partial q_2}P = 0.$$

That is,

$$45 - 2q_1 - 5 - q_2 = 0, \qquad 55 - 2q_2 - 5 - q_1 = 0.$$

That is, $q_2 = 40 - 2q_1$ and $q_1 = 50 - 2q_2$, so that $q_1 = 50 - 2(40 - 2q_1)$, so $q_1 = 50 - 80 + 4q_1$, so $3q_1 = 30$, and $q_1 = 10$. Then $q_2 = 40 - 2q_1 = 40 - 20 = 20$. So, the only critical point is

$$(q_1, q_2) = (10, 20).$$

(c) Use the second derivative test to determine if your critical points represent local maximums, local minimums, or saddle points.

We have

$$\frac{\partial^2}{\partial q_1^2}P = -2, \qquad \frac{\partial^2}{\partial q_2^2}P = -2, \qquad \frac{\partial^2}{\partial q_1 \partial q_2}P = -1.$$

So,

$$\frac{\partial^2}{\partial q_1^2}P \cdot \frac{\partial^2}{\partial q_2^2}P - \Big(\frac{\partial^2}{\partial q_1 \partial q_2}P\Big)^2 = (-2)(-2) - (-1)^2 = 4 - 1 = 3 > 0.$$

The second derivative test says we must then have a local max or min. Since $\frac{\partial^2}{\partial q_1^2}P < 0$, we must have a local max. So, $(10, 20)$ is a local maximum of P.

10. Question 10

Let f be the function $f(x, y) = 5x^2y^3$.

Find the average value of f on the rectangle defined by $0 \le x \le 3$ and $-4 \le y \le 4$.

Solution. The average value is the integral of f divided by the area of the region, which is $3 \cdot 8 = 24$. So, the average value is

$$\begin{aligned}\frac{1}{24}\int_{x=0}^{x=3}\int_{y=-4}^{y=4} 5x^2y^3\,dy\,dx &= \frac{1}{24}\int_{x=0}^{x=3} 5x^2\Big(\int_{y=-4}^{y=4} y^3\,dy\Big)dx = \frac{1}{24}\int_{x=0}^{x=3} 5x^2\Big([y^4/4]_{y=-4}^{y=4}\Big)dx \\ &= \frac{1}{24}\int_{x=0}^{x=3} 5x^2\Big(4^4/4 - (-4)^4/4\Big)dx = \frac{1}{24}\int_{x=0}^{x=3} 5x^2(0)\,dx = 0.\end{aligned}$$

11. Question 11

Let T be the triangle having vertices at $(0,0),(3,0)$, and $(3,6)$. Evaluate the integral of $f(x,y)=x^2y$ over the region bounded by the triangle T.

Solution. The triangle has hypotenuse lying inside the line $y=2x$. So, we can first integrate from $y=0$ to $y=2x$, and then integrate from $x=0$ to $x=3$. That is,

$$\begin{aligned}\iint_T f(x,y)dxdy &= \int_{x=0}^{x=3}\int_{y=0}^{y=2x} x^2ydydx = \int_{x=0}^{x=3} x^2\Big(\int_{y=0}^{y=2x} ydy\Big)dx\\ &= \int_{x=0}^{x=3} x^2\Big([(1/2)y^2]_{y=0}^{y=2x}\Big)dx = \int_{x=0}^{x=3} x^2(1/2)(2x)^2dx = \int_{x=0}^{x=3} 2x^4dx\\ &= (2/5)[x^5]_{x=0}^{x=3} = (2/5)3^5.\end{aligned}$$