118 Midterm 1 Solutions[1]

1. Question 1

Find the minimum and maximum values of

$$f(x) = x^3(1-x)^4$$

on the interval $[0,1]$.

Solution. We have $f'(x) = x^3 4(1-x)^3(-1) + (1-x)^4 3x^2 = x^2(1-x)^3[-4x+3(1-x)] = x^2(1-x)^3[3-7x]$. If $f'(x)=0$, we have $x=0$ or $x=1$ or $x=3/7$. These are the only critical points, and also $x=0,1$ are the endpoints of the interval. So, the minimum and maximum values must occur among the points: $0, 3/7, 1$. We check that $f(0)=0$, $f(1)=0$ and $f(3/7)>0$. So, the minimum value of f on $[0,1]$ is 0 (occurring at $x=0$ and $x=1$) and the maximum value of f is $(3/7)^3(4/7)^4$ occurring at $x=3/7$.

2. Question 2

(a) Find the length of the following vector

$$(1,2,3) + 2\cdot(2,3,5)$$

Solution. $\|(1,2,3)+(4,6,10)\| = \|(5,8,13)\| = \sqrt{25+64+169} = \sqrt{258}$.

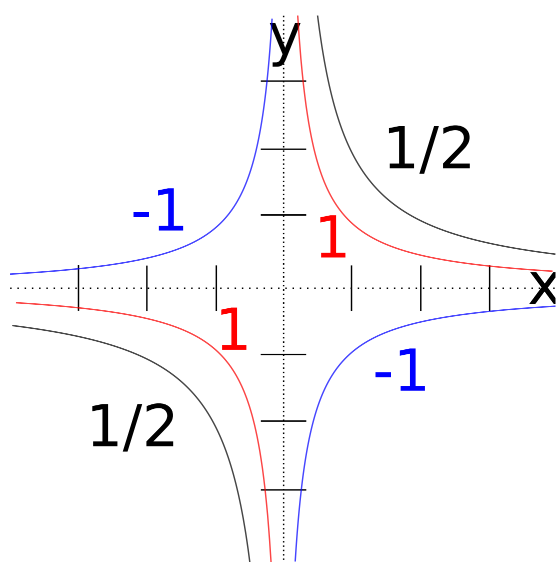
(b) Sketch the function

$$f(x,y) = \frac{1}{xy}$$

using a contour plot. **Only plot the contours $f(x,y)=-1$, $f(x,y)=1$ and $f(x,y)=\frac{1}{2}$.** Label each contour with the value that f takes on that contour.

3. Question 3

The following table summarizes some data about a function $f\colon \mathbf{R}\to\mathbf{R}$. We assume that f' and f'' exist and are continuous on all of $\mathbf{R}$. We list several points $x\in\mathbf{R}$, and we also list the values of: $f'(x)$, $f''(x)$. Using the following table, identify all of the listed local maxima, local minima, and inflection points, by writing an X in the corresponding column of the table.



If the point cannot be identified as a local extremum using the data at hand, and if the point cannot be identified as an inflection point with the data at hand, write an X in the column labelled "unknown." Also, if you know for sure that the point is not a local extremum and this point is not an inflection point, write an X in the column labelled "unknown."

It is also given information that $f''(x)>0$ on the interval $(5,7)$ and $f''(x)<0$ on the interval $(7,8)$, and $f''(x)>0$ on the interval $(8,10)$.

You do not need to show any work for this question.

[1]November 3, 2018,

x	$f'(x)$	$f''(x)$	local max	local min	inflection point	unknown
1	0	1		X		
2	0	0				X
3	1	0				X
4	0	-3	X			
5	-1	2				X
6	0	2		X		
7	1	0			X	
8	0	0			X	
9	1	2				X
10	1	0				X

Points 1,2,4 and 6 use the second derivative test for critical points. Points 3 and 5 are not a critical points, and there is not enough information to determine whether or not they are inflection points. Points 7 and 8 have a sign change of the second derivative, so they are inflection points. (Even though point 8 is a critical point, we cannot determine whether or not it is a local extremum; for example, $f(x) = x^3$ has a critical point at $x = 0$ which is an inflection point, but which is not a local extremum.) Point 9 is neither a critical point nor an inflection point. Point 10 is not a critical point, and there is not enough information to check whether or not it is an inflection point.

4. Question 4

Compute the following integrals.

(a) $\int_1^3 (x^{-2} + e^{2x})dx$.

Solution. $\int_1^3 (x^{-2} + e^{2x})dx = [-x^{-1} + (1/2)e^{2x}]_{x=1}^{x=3} = -(1/3) + (1/2)e^6 + 1 - (1/2)e^2 = (2/3) + (1/2)(e^6 - e^2)$.

(b) $\int x\ln(x)dx$.

Solution. We use $\int udv = uv - \int vdu$ where $u = \ln x$, $v = x^2/2$ so that $dv = xdx$ and $du = (1/x)dx$, so that

$$\int x\ln(x)dx = \int dvu = uv - \int vdu = (x^2/2)\ln x - \int (x^2/2)(1/x)dx$$
$$= (x^2/2)\ln x - \int (x/2)dx = (x^2/2)\ln x - x^2/4 + C.$$

5. Question 5

Compute the following integrals.

(a) $\int t^3(t^4+1)^5 dt$.

Solution. Substituting $u = t^4 + 1$ so that $du = 4t^3dt$, we have

$$\int_0^1 t^3(t^4+1)^6 dt = \int_{u(0)}^{u(1)} (1/4)u^6 du = \int_1^2 (1/4)u^6 du = (1/28)[u^7]_{u=1}^{u=2} = \frac{2^7 - 1}{28}.$$

(b) $\int_{-2}^2 t^{77}e^{t^4}dt$.

Solution. The function is odd so the integral on $[-2,2]$ must be zero. That is, if $f(t) = t^{77}e^{t^4}$, then $f(-t) = -f(t)$, i.e. $-f(-t) = f(t)$. So

$$\int_{-2}^2 f(t)dt = \int_{-2}^0 f(t)dt + \int_0^2 f(t)dt.$$

Substituting the first integral $u = -t$ so that $du = -dt$, we get

$$\int_{-2}^2 f(t)dt = \int_2^0 f(-u)(-du) + \int_0^2 f(t)dt = \int_2^0 f(u)du + \int_0^2 f(t)dt$$
$$= -\int_0^2 f(u)du + \int_0^2 f(t)dt = 0.$$