118 Midterm 2 Solutions[1]

1. Question 1

Find the minimum and maximum values of $f(x) = 2\sqrt{x^2+1} - x$ on the interval $[0, 2]$

Solution. We have $f'(x) = 2x(x^2+1)^{-1/2} - 1$. If $f'(x) = 0$, we have $2x(x^2+1)^{-1/2} = 1$, so that $2x = (x^2+1)^{1/2}$, and $4x^2 = x^2+1$, i.e. $3x^2 = 1$, i.e. $x = \pm 1/\sqrt{3}$. Since we are only considering $x \in [0, 2]$, we see that $x = 1/\sqrt{3}$ is the only critical point of f. So, the minimum and maximum values must occur among the points: $0, 1/\sqrt{3}, 2$. We check that $f(0) = 2$, $f(2) = 2\sqrt{5} - 2 = 2(\sqrt{5} - 1)$ and $f(1/\sqrt{3}) = 2\sqrt{1/3+1} - 1/\sqrt{3} = 3/\sqrt{3} = \sqrt{3}$. Since $\sqrt{3} < 2 < 2(\sqrt{5} - 1)$, the minimum value of f is $\sqrt{3}$ and the maximum value of f is $2(\sqrt{5} - 1)$.

2. Question 2

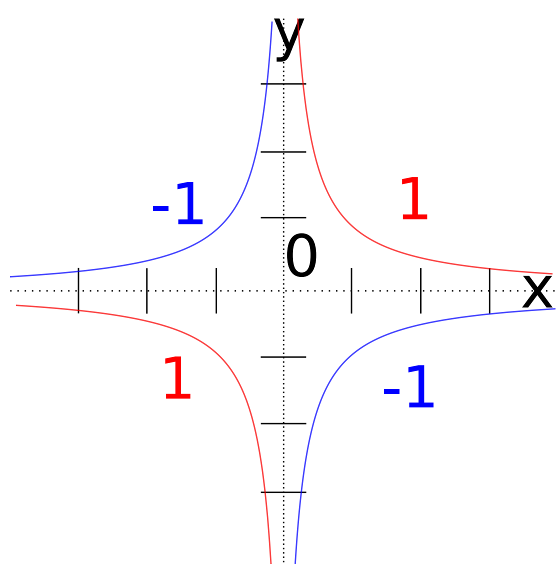
(a) Find a unit vector pointing in the same direction as $(1, 2, 4)$.

Solution. $\frac{(1,2,4)}{\|(1,2,4)\|} = \frac{1}{\sqrt{1+4+16}}(1, 2, 4) = \frac{1}{\sqrt{21}}(1, 2, 4)$.

(b) Sketch the function

$$f(x, y) = xy$$

using a contour plot. **Only plot the contours $f(x, y) = 0$, $f(x, y) = 1$ and $f(x, y) = -1$.** Label each contour with the value that f takes on that contour.



3. Question 3

The following table summarizes some data about a function $f : \mathbf{R} \to \mathbf{R}$. We assume that f' and f'' exist and are continuous on all of $\mathbf{R}$. We list several points $x \in \mathbf{R}$, and we also list the values of: $f'(x)$, $f''(x)$. Using the following table, identify all of the listed local maxima, local minima, and inflection points, by writing an X in the corresponding column of the table.

If the point cannot be identified as a local extremum using the data at hand, and if the point cannot be identified as an inflection point with the data at hand, write an X in the column labelled "unknown." Also, if you know for sure that the point is not a local extremum and this point is not an inflection point, write an X in the column labelled "unknown."

It is also given information that $f''(x) > 0$ on the interval $(5, 7)$ and $f''(x) < 0$ on the interval $(7, 8)$, and $f''(x) > 0$ on the interval $(8, 10)$.

You do not need to show any work for this question.

[1]November 3, 2018,

x	$f'(x)$	$f''(x)$	local max	local min	inflection point	unknown
1	0	1		X		
2	0	0				X
3	1	0				X
4	0	-3	X			
5	-1	2				X
6	0	2		X		
7	1	0			X	
8	0	0			X	
9	1	2				X
10	1	0				X

Points 1,2,4 and 6 use the second derivative test for critical points. Points 3 and 5 are not a critical points, and there is not enough information to determine whether or not they are inflection points. Points 7 and 8 have a sign change of the second derivative, so they are inflection points. (Even though point 8 is a critical point, we cannot determine whether or not it is a local extremum; for example, $f(x) = x^3$ has a critical point at $x = 0$ which is an inflection point, but which is not a local extremum.) Point 9 is neither a critical point nor an inflection point. Point 10 is not a critical point, and there is not enough information to check whether or not it is an inflection point.

4. Question 4

Compute the following integrals.
(a) $\int_1^3 (x^2 + x^{-2})dx$.
Solution. $\int_1^3 (x^2 + x^{-2})dx = [(1/3)x^3 - x^{-1}]_{x=1}^{x=3} = (1/3)3^3 - 1/3 - 1/3 + 1 = 10 - 2/3 = 28/3$.
(b) $\int xe^x dx$.
Solution. We use $\int udv = uv - \int vdu$ where $v = e^x$, $u = x$ so that $dv = e^x dx$ and $du = dx$, so that

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C.$$

5. Question 5

Compute the following integrals.
(a) $\int te^{t^2} dt$.
Solution. Substituting $u = t^2$ so that $du = 2tdt$, we have

$$\int te^{t^2} dt = \int e^u (1/2) du = (1/2)e^u + C = (1/2)e^{t^2} + C.$$

(b) $\int_{-2}^2 t^{99} e^{t^4} dt$.
Solution. The function is odd so the integral on $[-2, 2]$ must be zero. That is, if $f(t) = t^{99}e^{t^4}$, then $f(-t) = -f(t)$, i.e. $-f(-t) = f(t)$. So

$$\int_{-2}^2 f(t)dt = \int_{-2}^0 f(t)dt + \int_0^2 f(t)dt.$$

Substituting the first integral $u = -t$ so that $du = -dt$, we get

$$\int_{-2}^2 f(t)dt = \int_2^0 f(-u)(-du) + \int_0^2 f(t)dt = \int_2^0 f(u)du + \int_0^2 f(t)dt$$
$$= -\int_0^2 f(u)du + \int_0^2 f(t)dt = 0.$$