118 Midterm 1 Solutions[1]

1. Question 1

(a) Evaluate the following limit. If the limit does not exist, write DNE.

$$\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2+x}\right).$$

Solution.

$$\sqrt{x^2+x+1}-\sqrt{x^2+x}=(\sqrt{x^2+x+1}-\sqrt{x^2+x})\frac{\sqrt{x^2+x+1}+\sqrt{x^2+x}}{\sqrt{x^2+x+1}+\sqrt{x^2+x}}$$
$$=\frac{(x^2+x+1)-(x^2+x)}{\sqrt{x^2+x+1}+\sqrt{x^2+x}}=\frac{1}{\sqrt{x^2+x+1}+\sqrt{x^2+x}}.$$

So, letting $x\to\infty$, we get

$$\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2+x}\right)=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+x+1}+\sqrt{x^2+x}}=0.$$

The last line used that $\lim_{x\to\infty}\sqrt{x^2+x+1}+\sqrt{x^2+x}=\infty$.

(b) Is there a real number a such that the following limit exists?

$$\lim_{x\to1}\frac{3x^2+ax+a+1}{(x-1)(x+1)}$$

If so, find the value of a and the value of the limit.

Solution. If we plug in 1 into the denominator, we get 0. So, in order for the limit to exist, we need to get 0 when we plug in 1 into the numerator. That is, we must have $3+a+a+1=0$. That is, $a=-2$. To see that the limit exists, note that $\lim_{x\to1}\frac{3x^2+ax+a+1}{(x-1)(x+1)}=\lim_{x\to1}\frac{3x^2-2x-1}{(x-1)(x+1)}=\lim_{x\to1}\frac{(x-1)(3x+1)}{(x-1)(x+1)}=\lim_{x\to1}\frac{(3x+1)}{(x+1)}=\frac{3+1}{1+1}=2.$. The first few equalities use algebra. The penultimate equality uses the limit law for quotients, which allows us to plug in the value $x=1$ into $\frac{(3x+1)}{(x+1)}$. Alternatively, note that $\frac{(3x+1)}{(x+1)}$ is continuous at $x=1$, so the limit at $x=1$ is equal to the value of $\frac{(3x+1)}{(x+1)}$ at $x=1$.

2. Question 2

Label the following statements as TRUE or FALSE. Then, briefly **justify your answer**.

(a) If a function is continuous at 0, then $f'(0)$ exists.

FALSE. As shown in class, when $f(x)=|x|$, f is continuous but $f'(0)$ does not exist.

(b) Let $f\colon\mathbf{R}\to\mathbf{R}$. If $f(0)=-1$ and $f(1)=1$, then there is some $x\in[0,1]$ such that $f(x)=0$.

FALSE. We can define f to only take values -1 and 1, so f would then never be zero. (We did not assume that f is continuous, so the Intermediate Value Theorem does not apply.)

(c) If $b>0$, then $\frac{d}{dx}b^x=b^x$.

FALSE. $\frac{d}{dx}b^x=b^x\ln b$. So, for example, $(d/dx)2^x=2^x\ln 2\neq 2^x$.

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3. Question 3

Find the equation of all tangent lines to the curve $y = e^{-x^2}$ that are parallel to the line $y = 3$.

Solution. The tangent line is given by $y = 1$

From the chain rule, we have $y'(x) = -2xe^{-x^2}$. (If $f(x) = e^x$ and $g(x) = -x^2$ then $f'(x) = e^x$, $g'(x) = -2x$, and $y'(x) = (d/dx)f(g(x)) = f'(g(x))g'(x) = e^{-x^2}(-2x)$.)

In order to be tangent to the line $y = 3$, the tangent line must have slope 0, i.e. we should solve $y'(x) = 0$. Since the exponential function is always positive, the equation $-2xe^{-x^2} = 0$ reduces to $x = 0$. In the case $x = 0$, we have $y(0) = e^0 = 1$ and $y'(0) = 0$. So, the only tangent line with slope 0 is the line $y = 1$.

4. Question 4

Let $f(x) = x^3$. Using the **definition of the derivative**, show that

$$f'(x) = 3x^2.$$

Solution

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && \text{, by the definition of the derivative} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} && \text{, by the definition of the } f \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} && \text{, by algebra} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} && \text{, canceling the } x^3 \text{ terms} \\
&= \lim_{h\to 0} 3x^2 + 3xh + h^2 && \text{, canceling the } h \text{ terms} \\
&= 3x^2 && \text{, plugging in } h = 0.
\end{aligned}$$

5. Question 5

Let f be a function with $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 9$ and $f'(3) = 6$. Let

$$F(x) = \frac{1}{f(xf(x))}.$$

Find $F'(1)$.

Solution. $F'(1) = -6$. To see this, observe that

$$\begin{aligned}
F'(x) &= \frac{-\frac{d}{dx} f(xf(x))}{[f(xf(x))]^2}, && \text{quotient rule} \\
&= \frac{-f'(xf(x))\frac{d}{dx}(xf(x))}{[f(xf(x))]^2}, && \text{chain rule} \\
&= \frac{-f'(xf(x))(xf'(x) + f(x))}{[f(xf(x))]^2}, && \text{product rule.}
\end{aligned}$$

So, plugging in the given values of f and its derivatives,

$$F'(1) = \frac{-f'(f(1))(f'(1) + f(1))}{[f(f(1))]^2} = \frac{-f'(2)(4+2)}{[f(2)]^2} = \frac{-9(6)}{[3]^2} = \frac{-9(6)}{9} = -6.$$