118 Midterm 1 Solutions[1]

1. Question 1

(a) Evaluate the following limit. If the limit does not exist, write DNE.

$$\lim_{x\to 0}\Big(\frac{1}{x}-\frac{1}{x^2+x}\Big).$$

Solution. $\lim_{x\to 0}\Big(\frac{1}{x}-\frac{1}{x^2+x}\Big)=\lim_{x\to 0}\Big(\frac{x+1}{x^2+x}-\frac{1}{x^2+x}\Big)=\lim_{x\to 0}\Big(\frac{x}{x^2+x}\Big)=\lim_{x\to 0}\Big(\frac{1}{x+1}\Big)=\frac{1}{1}=1$. The first few equalities use algebra. The penultimate equality uses the limit law for quotients, which allows us to plug in the value $x=0$ into $1/(x+1)$. Alternatively, note that $1/(x+1)$ is continuous at $x=0$, so the limit at $x=0$ is equal to the value of $1/(x+1)$ at $x=0$.

(b) Find all values of the constant a such that

$$\lim_{x\to 0}\frac{\sqrt{ax+4}-2}{x}=1.$$

Solution. $\lim_{x\to 0}\frac{\sqrt{ax+4}-2}{x}=\lim_{x\to 0}\frac{\sqrt{ax+4}-2}{x}\frac{\sqrt{ax+4}+2}{\sqrt{ax+4}+2}=\lim_{x\to 0}\frac{ax+4-4}{x(\sqrt{ax+4}+2)}=\lim_{x\to 0}\frac{ax}{x(\sqrt{ax+4}+2)}=\lim_{x\to 0}\frac{a}{(\sqrt{ax+4}+2)}=\frac{a}{(\sqrt{4}+2)}=\frac{a}{4}$. The first few equalities use algebra. The penultimate equality uses the limit law for quotients, which allows us to plug in the value $x=0$ into $\frac{a}{(\sqrt{ax+4}+2)}$. Alternatively, note that $\frac{a}{(\sqrt{ax+4}+2)}$ is continuous at $x=0$, so the limit at $x=0$ is equal to the value of $\frac{a}{(\sqrt{ax+4}+2)}$ at $x=0$.

So, we must have $a=4$.

2. Question 2

Compute

$$\lim_{x\to 0}\frac{|2x-1|-|2x+1|}{x}.$$

If the limit does not exist, write DNE. *Solution.* Let x with $|x|<1/2$. Then $|2x|<1$, so $2x-1\leq|2x|-1<0$. In particular, $|2x-1|=1-2x$. Consider again x with $|x|<1/2$. Then $|2x|<1$, so $2x+1\geq-|2x|+1>0$. In particular, $|2x+1|=2x+1$. Combining these two facts, we have, for $|x|<1/2$

$$\frac{|2x-1|-|2x+1|}{x}=\frac{1-2x-(2x+1)}{x}=\frac{-4x}{x}=-4.$$

As $x\to 0$, we only need to consider $|x|<1/2$, so we conclude that

$$\lim_{x\to 0}\frac{|2x-1|-|2x+1|}{x}=\lim_{x\to 0}(-4)=-4.$$

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3. Question 3

Find the equation of the tangent lines to the curve $y = \dfrac{x-1}{x+1}$ that are parallel to the line $x - 2y = 2$.

Solution. The tangent lines are given by $y = (1/2)x - 1/2$ and $y = (1/2)x + 7/2$.

Let $x \neq -1$. Then $y'(x) = \frac{(x+1)-(x-1)}{(x+1)^2} = 2/(x+1)^2$. The line $x - 2y = 2$ has slope $1/2$, so we solve the equation $y'(x) = 1/2$. We get $(x+1)^2 = 4$, so $|x+1| = 2$, so $x = 1, -3$. Note that $y(1) = 0$ and $y(-3) = 2$. So, the two tangent lines are $y = (1/2)(x-1)$ and $y = (1/2)(x+3) + 2$.

4. Question 4

For the following functions, determine whether or not $f'(0)$ exists. If $f'(0)$ exists, compute its value.

(a) f is the inverse of the natural logarithm function.
(b) $f(x) = x^{1/3}$.

Solution of (a) We have $f(x) = e^x$, and we know from class that $f'(x) = e^x$, so $f'(0) = e^0 = 1$.

Solution of (b) We have

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{h^{1/3}}{h} = \lim_{h\to 0} h^{-2/3}.$$

The quantity $h^{-2/3}$ becomes arbitrarily large as $h \to 0$ since $h^{2/3}$ goes to zero as $h \to 0$, so the limit does not exist, i.e. $f'(0)$ DNE.

5. Question 5

Let f be a function such that $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$ and $f'(3) = 6$. Define

$$F(x) = f(xf(xf(x))).$$

Find $F'(1)$.

Solution. $F'(1) = 198$. To see this, observe that

$$\begin{aligned}
F'(x) &= f'(xf(xf(x))) \cdot \frac{d}{dx}[xf(xf(x))], && \text{chain rule}\\
&= f'(xf(xf(x))) \cdot \left(x\frac{d}{dx}[f(xf(x))] + f(xf(x)) \right), && \text{product rule}\\
&= f'(xf(xf(x))) \cdot \left(x(f'(xf(x)) \cdot \frac{d}{dx}[xf(x)]) + f(xf(x)) \right), && \text{chain rule}\\
&= f'(xf(xf(x))) \cdot \left[x(f'(xf(x)) \cdot [xf'(x) + f(x)]) + f(xf(x)) \right], && \text{product rule}
\end{aligned}$$

So, plugging in the given values of f and its derivatives,

$$\begin{aligned}
F'(1) &= f'(f(f(1))) \cdot [f'(f(1)) \cdot [f'(1) + f(1)] + f(f(1))]\\
&= f'(f(2)) \cdot [f'(2) \cdot [4+2] + f(2)] = f'(3) \cdot [5 \cdot 6 + 3] = 6 \cdot [33] = 198
\end{aligned}$$