Research Statement, Steven Heilman

I am interested in applying analytic techniques to probabilistic problems. These problems are sometimes motivated by hardness results in theoretical computer science. A recurring theme of our research is that mathematical objects that were initially motivated by physics (such as semigroups of operators, hypercontractivity, Fourier analysis, soap bubbles and minimal surfaces, Bell's inequality from quantum mechanics, optimization of energy functionals, the calculus of variations, etc.) now have renewed motivation from important problems in computer science.

For example, one main focus of my research has been isoperimetric problems in Euclidean space equipped with the Gaussian measure [HJN13, Hei14, HMN16, Hei21b, Hei20b, Hei21d, Hei21c, Hei19, HT21, Hei21e, HT22, Hei22a, Hei22b, Hei23b, Hei23a]. As an application [Hei23b], we recently proved that the three candidate plurality voting method is the "democratic" voting method that best preserves an election's outcome when votes have been randomly corrupted, when the correlation of the original and corrupted votes is in the range $[-1/43, 1/10]$. This proves part of the Plurality is Stablest Conjecture of [KKMO07] from 2005, improving on our more limited prior result [HT21]. Since our paper [Hei23b] applies for negative correlations, we deduce a new computational hardness result for approximately solving the MAX-3-CUT problem, improving on a previous similar result from 1997 [KKLP97].

In [Hei23a], we proved a sphere-valued inequality analogous to the one proven in [Hei23b], proving part of a conjecture of [HNP$^+$21]. We then deduce a new computational hardness result for approximately solving the Quantum MAX-CUT problem. This computational hardness result is conditional on the Unique Games Conjecture.

We previously adapted similar tools to prove that a finite number of samples suffice for non-interactive simulation from correlated Gaussian sources [HT22]. We also formulated a version of the Plurality is Stablest Conjecture for ranked choice voting [Hei22c], and extended the result of [HT21] to that setting.

Recently [Hei21d, Hei22a], I also made significant progress on a question of Barthe from 2001 [Bar01]. This question asks for the symmetric Euclidean set of fixed Gaussian volume with smallest Gaussian surface area. The work [Hei22a] answers this question, with a small range of exceptions.

In [Hei21e], I proved some cases of a conjecture of Eldan [Eld15], thereby giving a variational proof of a robust version of Borell's inequality. That is, if a Euclidean set is close to maximizing its noise stability, then that set is close to a half space. The main result of [Hei21e] combined with previous works such as [BBJ17, Hei19, HT21] essentially shows that one single argument can prove nearly every known inequality for sets or partitions that maximize noise stability, with respect to Gaussian volume constraints. So, instead of having disparate arguments to prove these inequalities, one single calculus of variations argument has emerged, providing an aesthetically pleasing way to prove these optimal inequalities.

Recently, I also proved a stability version of the Gaussian multi-bubble problem [Hei19]. This question asks for the partition of Euclidean space into $k \geq 3$ sets of fixed Gaussian volumes with smallest total Gaussian surface area. I showed that if a partition has nearly minimal Gaussian surface area, then this partition is close to the optimal partition. This result [Hei19] improves upon the solution of the Gaussian multi-bubble problem in [MN22a], where the optimal partition is identified. Also, [Hei19] implies previously unobtained applications to voting [IM12, Hei21a]. The results [MN22a, Hei19] were preceded by a sequence of works including [MN22a, Hei21c] where $k = 3$ and $k = 4$ were treated, respectively. This problem was open since at least the 1990s [SM96, Problem 2] [Hut97]. The case $k = 2$ has been solved since the 1970s [SC74]. The solution of the "double-bubble problem" for the Lebesgue measure, i.e. finding two disjoint Euclidean sets with minimal total Euclidean surface area was resolved in a well-known 2002 result [HMRR02]. The case of at most six sets was resolved in [MN22b, MN23], though more than six sets remains elusive.

I also solved [Hei20b] the endpoint case of a conjecture of Khot and Moshkovitz related to the Unique Games Conjecture, less a small error. The Unique Games Conjecture is a contemporary proxy for the P versus NP problem. So, my result [Hei20b] gives indirect evidence for an important conjecture in computational complexity theory.

A key aspect of the results [Hei21d, Hei20b, Hei21c, Hei19, HT21, Hei21e, HT22, Hei22a, Hei23b, Hei23a] is that they are dimension independent. That is, the proof techniques and results hold simultaneously for Euclidean space of any dimension.

My long term goal is to develop general methods for approaching a wide class of Gaussian isoperimetric problems, since such general methods did not exist previously. Most recently, I have been developing calculus of variations techniques [Hei20b, Hei21c, HT21, Hei21e, HT22, Hei22a, Hei22b]. These techniques previously had little to no usage in this area.

Currently, there are many unsolved Gaussian isoperimetric problems, and their resolution will yield countless dividends. The Gaussian measure is most interesting since it is almost interchangeable with the uniform measure on the discrete hypercube, via Central Limit Theorems [Rot79, Cha06, MOO10, Mos10, IM12]. These Gaussian isoperimetric results therefore imply inequalities on the discrete hypercube. The discrete inequalities are then applied to machine learning and Grothendieck inequalities [KN09, KN13], to the Unique Games Conjecture [KKMO07, MOO10, KM16], to semidefinite programming algorithms such as MAX-CUT [KKMO07, IM12], to social choice theory [MOO10, IM12, Hei21a], to learning theory [FGRW12], to communication complexity [CR11], to the noninteractive simulation problem [DMN17, DMN18, GKR18, HT22], etc. So, solving these isoperimetric problems can tell us how quickly computers can run, and how to design elections so that erroneous tabulation of votes (or hacking) does not affect the outcome of the election.

I have also worked on independent sets in random graphs and random trees [Hei20a, Hei22d]. An independent set of size k in a finite undirected graph is a set of k vertices of the graph, no two of which are connected by an edge. Let $x_k(G)$ be the number of independent sets of size k in a graph G and let $\alpha(G) = \max\{k \geq 0 \colon x_k(G) \neq 0\}$. In 1987, Alavi, Malde, Schwenk and Erdős asked if the independent set sequence $x_0(G), x_1(G), \ldots, x_{\alpha(G)}(G)$ of a tree is unimodal (the sequence goes up and then down). This problem is still open. In 2006, Levit and Mandrescu showed that the last third of the independent set sequence of a tree is decreasing. We show [Hei20a, Hei22d] that the first 46.8% of the independent set sequence of a random tree is increasing with (exponentially) high probability as the number of vertices goes to infinity. So, the question of Alavi, Malde, Schwenk and Erdős is "four-fifths true", with high probability.

We also show unimodality of the independent set sequence of Erdős–Rényi random graphs, when the expected degree of a single vertex is large (with (exponentially) high probability as the number of vertices in the graph goes to infinity, except for a small region near the mode).

I have also studied L_p Poincaré inequalities for $1 < p < \infty$ [HMO14] on general measure spaces that do not seem to be provable using standard techniques, such as Littlewood-Paley theory. In particular, we proved the degree one case of a conjecture of [MN14]. The conjectured Poincaré inequalities [MN14] can be used to construct graphs that are expanders in a very general sense [MN14]. The explicit construction of expander graphs are then used in computer science [HLW06].

1. The Standard Simplex Conjecture

1.1. **Euclidean Isoperimetry.** Classical isoperimetry can be traced to ancient times, though its full understanding still remains incomplete. Generally speaking, we look for an object with least surface area among all objects of fixed volume. And we expect that the smallest surface area object has a simple structure.

Isoperimetric problems and minimal surface theory have a long history in differential geometry [Ste38, Sch72, Min96, Wei27, Hur02, Lév51, Sim68, Law70, Bor75, Alm76, Gro83, Sim83, BdC84, EH89, Tal95, BL96, HMRR02, CM12, CIMW13], with some motivation from the physics of soap bubbles [Pla73, Tay76]. In particular, soap bubbles we encounter in the real world are minimal surfaces. Many deep results have come from these investigations, and we hope to continue this tradition.

To see that our knowledge is still limited, note that, until 2022, we did not know the three disjoint sets of fixed Euclidean volume with minimum total surface area. This problem is known as a triple bubble, or multi bubble problem [HMRR02, Rei08, CCH+08]. Recently, I essentially solved the Gaussian version of this problem [Hei21c]. The Gaussian case for an arbitrary number of sets was then resolved in [MN22a], and the Euclidean case for at most 5 sets was resolved in [MN22b]. A generalization of this problem, known as the Standard Simplex Conjecture, is still open (see (8) below). I am currently working on this more general problem [Hei19, Hei21e]. Gaussian isoperimetric problems have gained recent interest due to their applications in theoretical computer science (see Section 1.5). For mathematical background motivation, we begin with the usual Euclidean isoperimetric inequality.

Let $n \geq 1$ be an integer, let $A \subseteq \mathbb{R}^n$ be a Borel set with smooth boundary ∂A. Let $\mathrm{vol}_n(A)$ denote the Euclidean volume of A, and let $\mathrm{vol}_{n-1}(\partial A)$ denote the Euclidean surface area of ∂A. Let $r > 0$ and let $B(0,r) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq r^2\}$ be a Euclidean ball such that $\mathrm{vol}_n(A) = \mathrm{vol}_n(B(0,r))$. The **Classical Isoperimetric Inequality** says that the Euclidean ball has the smallest boundary among all sets of fixed volume:

$$\mathrm{vol}_n(A) = \mathrm{vol}_n(B(0,r)) \quad \Longrightarrow \quad \mathrm{vol}_{n-1}(\partial A) \geq \mathrm{vol}_{n-1}(\partial B(0,r)). \tag{1}$$

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and recall that $\|x\|_2 := (x_1^2 + \cdots + x_n^2)^{1/2} = \langle x, x\rangle^{1/2}$. Let dx be Lebesgue measure on $\mathbb{R}^n$, and let $\gamma_n(x) := e^{-\|x\|_2^2/2}(2\pi)^{-n/2}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded (measurable) function and let $t \geq 0$. Let $\Delta := -\sum_{i=1}^n \partial^2/\partial x_i^2$. Let $e^{-t\Delta} f(x)$ denote the **classical heat semigroup** applied to f. That is, for any $x \in \mathbb{R}^n$,

$$e^{-t\Delta} f(x) := \int_{\mathbb{R}^n} f(x + y\sqrt{2t}\,) d\gamma_n(y).$$

The classical isoperimetric inequality (1) is a consequence of the following inequality for the heat semigroup $e^{-t\Delta}$, which can be proven via symmetrization [Led96].

$$\mathrm{vol}_n(A) = \mathrm{vol}_n(B(0,r)) \quad \Longrightarrow \quad \forall\, t \geq 0, \quad \int_{\mathbb{R}^n} 1_A \cdot e^{-t\Delta} 1_A dx \leq \int_{\mathbb{R}^n} 1_{B(0,r)} \cdot e^{-t\Delta} 1_{B(0,r)}\, dx. \tag{2}$$

So, as heat flows out of a given set, the one that preserves the most of its heat is the ball. To get (1) from (2), let $t \to 0$ in (2). The quantity $\int_{\mathbb{R}^n} 1_A \cdot e^{-t\Delta} 1_A dx$ is sometimes called the heat content of A. We will focus on statements of the form (2) below.

1.2. **Gaussian Isoperimetry.** Let $A \subseteq \mathbb{R}^n$ be a Borel set. Denote $\gamma_n(A) := \int_A \gamma_n(x) dx$. Let $H \subseteq \mathbb{R}^n$ be a **half space**. That is, H is the region that lies on one side of a hyperplane. The **Gaussian Isoperimetric Inequality** [SC74] says that the half space H has the smallest Gaussian surface area among all sets of fixed Gaussian volume, i.e.

$$\gamma_n(A) = \gamma_n(H) \quad \Longrightarrow \quad \gamma_{n-1}(\partial A) \geq \gamma_{n-1}(\partial H). \tag{3}$$

Here, we denote the **Gaussian surface area** of ∂A by

$$\gamma_{n-1}(\partial A) := \liminf_{\varepsilon \to 0}\, (2\varepsilon)^{-1} \gamma_n(x \in \mathbb{R}^n : \exists\, y \in \partial A, \|x - y\|_2 < \varepsilon).$$

The result (3) of [SC74] has been elucidated and strengthened over the years [Bor85, Led94, Led96, Bob97, BS01, Bor03, MN15a, MN15b, Eld15, MR15, BBJ17].

For applications to theoretical computer science, it is more useful to have a Gaussian version of (2). To this end, we first replace the heat semigroup by the Ornstein-Uhlenbeck semigroup. Define the operator L by $Lf(x) := \Delta f(x) + \langle x, \nabla f(x)\rangle$, for any $x \in \mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $t \geq 0$, define the **Ornstein-Uhlenbeck semigroup** applied to $f\colon \mathbb{R}^n \to \mathbb{R}$ by

$$e^{-tL}f(x) := \int_{\mathbb{R}^n} f(xe^{-t} + y\sqrt{1-e^{-2t}})d\gamma_n(y).$$

Then Borell's inequality [Bor85], which can be proven by symmetrization [BS01], says

$$\gamma_n(A) = \gamma_n(H) \quad \Longrightarrow \quad \forall\, t \geq 0, \quad \int_{\mathbb{R}^n} 1_A \cdot e^{-tL} 1_A d\gamma_n \leq \int_{\mathbb{R}^n} 1_H \cdot e^{-tL} 1_H d\gamma_n. \tag{4}$$

Borell's inequality [Bor85] also holds for negative correlations, with the inequality reversed:

$$\gamma_n(A) = \gamma_n(H) \quad \Longrightarrow \quad \forall\, t \geq 0, \quad \int_{\mathbb{R}^n} 1_A \cdot (-e^{-t})^L 1_A d\gamma_n \geq \int_{\mathbb{R}^n} 1_H \cdot (-e^{-t})^L 1_H d\gamma_n. \tag{5}$$

One can deduce (3) from (4), by letting $t \to 0^+$ in (4). The quantity $\int_{\mathbb{R}^n} 1_A \cdot e^{-tL} 1_A d\gamma_n$ is referred to as the **noise stability** of the set A with **correlation** e^{-t}. Borell's original result (4) from [Bor85] has been highly influential, and we wish to duplicate his success by generalizing (4). Inequality (4) should be considered well-understood, due to recent proofs [MN15b, Eld15] of stability versions of (4). That is, the inequality (4) is close to equality if and only if A is close to a half space.

1.3. **Differential Geometry and the Colding-Minicozzi Theory for Gaussian Minimal Surfaces.** In a landmark investigation of mean curvature flow [CM12], Colding and Minicozzi studied a maximal version of the Gaussian surface area of an $(n-1)$-dimensional hypersurface $\Sigma \subseteq \mathbb{R}^n$. They called this translation and dilation-invariant quantity

$$\sup_{a>0, b\in\mathbb{R}^n} \int_{\Sigma} a^{-\frac{n-1}{2}} \gamma_n((x-b)a^{-1/2})dx \tag{6}$$

the "entropy" of Σ. The Colding-Minicozzi entropy (6) is of interest since it monotonically decreases under the mean curvature flow. For this reason, [CM12] studied (local) minimizers of (6). In the context of mean curvature flow, the Colding-Minicozzi entropy (6) is an analogue of Perelman's reduced volume for Ricci flow.

As I demonstrated in [Hei21d, Hei21c], the methods of [CM12] also apply to the Gaussian surface area functional itself. Such a connection did not previously appear in the literature.

The Colding-Minicozzi theory studies eigenfunctions of the Ornstein-Uhlenbeck type operator

$$\Delta_\Sigma + \langle x, \nabla_\Sigma\rangle - \|F\|_2^2 - 1, \qquad \forall\, x \in \Sigma \subseteq \mathbb{R}^n, \tag{7}$$

associated to the surface $\Sigma := \partial A$. Here $\Delta_\Sigma, \nabla_\Sigma$ are the Laplacian and gradient, respectively, on the surface Σ. Also, $F = F_x$ is the **second fundamental form** of Σ at x, i.e. F is the matrix of first order partial derivatives of the unit normal vector at $x \in \Sigma$, and $\|F\|_2^2$ is the sum of the squares of the entries of F.

It was conjectured in [CIMW13] and ultimately proven using the Colding-Minicozzi theory in [Zhu20] that, among all compact $(n-1)$-dimensional hypersurfaces $\Sigma \subseteq \mathbb{R}^n$ with $\partial\Sigma = \emptyset$, the round sphere minimizes the quantity (6).

The idea of studying the eigenfunctions of an operator restricted to a minimal surface seems to be due to Simons [Sim68].

1.4. **Social Choice Theory.** By combining (4) with the invariance principle (16) below, the work [MOO10] solved the Majority is Stablest problem. This problem says that the most noise-stable way to determine the winner of an election between two candidates is to take the majority. This result assumes that no one person has too much influence over the election's outcome. The rigorous statement of this problem uses functions with domain $\{-1,1\}^n$ (see Section 9.1). The corresponding statement for more than two voters, the Plurality is Stablest Problem, would follow from Conjecture (8) below. For this reason, Conjecture (8) below is of interest. The Plurality is Stablest Problem says that the most noise-stable way to determine the winner of an election between $k > 2$ candidates is to take the plurality. This result assumes that no one person has too much influence over the election's outcome [IM12, Hei21a].

Applications of mathematics to the analysis of elections arguably began with Marquis de Condorcet in the 1700s, with further developments by Game Theorists such as Shapley, Shubik and Banzhaf in the 1950s and 1960s [SS54, Ban65]. In the last three decades, discrete Fourier analysis has provided new insights into social choice theory for mathematics and computer science [KKL88, Kal02, MOO10, Mos12]. Yet, the Plurality is Stablest Problem for elections with three or more candidates is still unresolved [IM12], since the three set version of (4) is still unresolved (see (8) below when $k = 3$). However, we recently solved this problem [HT21] in the case $k = 3$ when the correlation parameter e^{-t} is small.

1.5. **Computational Complexity of Clustering Algorithms: MAX-k-CUT.** Borell's inequality (4) gives a sharp computational hardness result for the MAX-CUT problem [KKMO07]. The MAX-CUT problem asks for the partition of the vertices of an undirected graph into two disjoint sets that maximizes the number of edges going between the two sets. One can find a partition of the vertices achieving .87856 times the maximum number of cut edges in polynomial time [GW95]. And assuming the Unique Games Conjecture, the constant .87856 is the largest possible number for which the previous sentence holds. When we modify the MAX-CUT problem to allow a partition of the vertices of the graph into $k > 2$ disjoint sets, we get the MAX-k-CUT problem. And for this problem, there is only a conjecture for the best possible approximation that can be done in polynomial time. More specifically, there is a polynomial time algorithm that finds a partition of vertices of some constant c_k times the maximum number of cut edges. And if (8) below is true, then the existence of a polynomial time algorithm achieving a guaranteed fraction of cut edges larger than c_k would violate the Unique Games Conjecture [IM12]. (We already know that $c_2 = .87856$ [KKMO07].) So, Conjecture (8) below tells us the best possible way to cluster graphs into different pieces. The MAX-k-CUT problem can be considered a clustering problem, since it allows data to be clustered into disjoint pieces (e.g. we could consider a two data points to be connected by an edge if we judge the data points to be dissimilar). Algorithms for MAX-k-CUT have also been applied to the community detection problem for graphs [AS15, ABKK17, AL18, HWX16, MPW16].

1.6. **Gaussian Isoperimetry for multiple sets.** The endpoint case of the Standard Simplex Conjecture asks for the minimum total Gaussian surface area of a partition of $\mathbb{R}^n$ into $k > 2$ sets, each of Gaussian measure $1/k$. The full conjecture, stated in (8), does not seem to follow from a symmetrization argument [BS01, IM12], or from the methods of [MN15b, Eld15]. However, the calculus of variations methods we have developed [Hei21d, Hei21c] do apply to this problem.

Let $A_1, \ldots, A_k \subseteq \mathbb{R}^n$ with $3 \leq k \leq n+1$, $n \geq 2$, $\cup_{i=1}^k A_i = \mathbb{R}^n$, $\gamma_n(A_i) = 1/k$. We now describe the conjectured maximizer of noise stability. Let $z_1, \ldots, z_k \in \mathbb{R}^n$ be the vertices of a regular k-simplex, which is centered at the origin of $\mathbb{R}^n$. For any $i = 1, \ldots, k$, let $B_i := \{x \in \mathbb{R}^n : \langle x, z_i\rangle = \max_{j=1,\ldots,k}\langle x, z_j\rangle\}$. Then $\{B_i\}_{i=1}^k$ is a partition of $\mathbb{R}^n$ into k regular simplicial cones. Generalizing (4), the **Standard Simplex Conjecture** says

$$\begin{aligned}
&\gamma_n(A_i) = 1/k,\, \forall\, i = 1, \ldots, k \quad \wedge \quad \cup_{i=1}^k A_i = \mathbb{R}^n \\
&\implies \quad \forall\, t \geq 0, \quad \sum_{i=1}^k \int_{\mathbb{R}^n} 1_{A_i} \cdot e^{-tL} 1_{A_i} d\gamma_n \leq \sum_{i=1}^k \int_{\mathbb{R}^n} 1_{B_i} \cdot e^{-tL} 1_{B_i} d\gamma_n.
\end{aligned} \tag{8}$$

Analogous to (5), the Standard Simplex Conjecture has a reversed inequality for negative correlations, and the volume constraint is removed:
(9)

$$\cup_{i=1}^{k} A_i = \mathbb{R}^n \quad \Longrightarrow \quad \forall\, t \geq \log(k-1), \quad \sum_{i=1}^{k} \int_{\mathbb{R}^n} 1_{A_i} \cdot (-e^{-t})^L 1_{A_i} d\gamma_n \geq \sum_{i=1}^{k} \int_{\mathbb{R}^n} 1_{B_i} \cdot (-e^{-t})^L 1_{B_i} d\gamma_n.$$

Morally speaking, the results of [Eva93] imply that, if equality holds in (8), then for all $i = 1, \ldots, k$, the set ∂A_i should have constant mean curvature, except on a negligible subset. It is difficult to turn this intuition into a proof (unless we know ahead of time that the same sets optimize noise stability in (8) for all $t > 0$ [MS02]), but this intuition explains why the Standard Simplex Conjecture (8) is believed to be true.

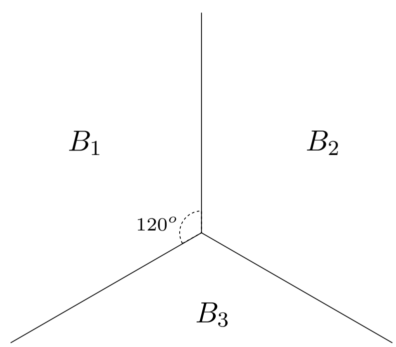

FIGURE 1. Optimal Sets for Conjecture (8) in the case $k = 3$, $n = 2$.

1.7. **Our Contribution.** We recently proved Conjecture (8) for $k = 3$ when $t > \log(10)$, and we also proved the negative correlation case (Conjecture 9) when $t > \log(43)$. The latter result implies that MAX-3-CUT cannot be solved in polynomial time within a multiplicative factor of .98937, assuming the Unique Games Conjecture. This result improves upon the known NP-hardness for MAX-3-CUT [KKLP97] from 1997, which says that MAX-3-CUT is NP-hard to approximate within a multiplicative factor of $1 - 1/(34(3)) \approx .990\ldots$.

Before [Hei23b, Hei20a], only the case $k = 2$ of Conjectures (8), (9) was known. As another corollary, we prove the Plurality is Stablest Conjecture for 3 candidate elections, for the same range of parameters t. The Plurality is Stablest Conjecture has been open since 2005 [KKMO07].

Earlier, we obtained a variational proof of Borell's Inequality (corresponding to the case $k = 2$, where (8) was previously known by different proof methods). In fact, the main result of [HT21] shows that sets optimizing the noise stability in Conjecture 8 must be $k - 1$ dimensional, in the sense that there exist (measurable) $A_1', \ldots, A_k' \subseteq \mathbb{R}^{k-1}$ such that, after applying a rotation to the sets $A_1, \ldots, A_k$, we have

$$A_i = A_i' \times \mathbb{R}^{n-k+1}, \qquad \forall\, 1 \leq i \leq k.$$

In [Hei22b], we conditionally solves Conjecture 8 for $k = 3$ or $k = 4$, with an extra assumption. That is, we put forward a potential strategy for resolving Conjecture 8.

I also formulated a conjecture [Hei22c] that Borda count is the ranked choice voting method that best preserves an election's outcome due to corruption or miscount. This result assumes that no one person has too much influence on the outcome of the election, and the voting method satisfies the Condorcet Loser Criterion. (The Condorcet Loser Criterion says that if one candidate loses to all other candidates in head-to-head comparisons, then that candidate cannot win the election.) The PI also showed [Hei22c] that this conjecture follows from the Plurality is Stablest Conjecture. Consequently, the partial result of [HT21] extends to the Borda count is Stablest conjecture.

Earlier, I proved a stability version of the endpoint case $t \to 0^+$ of Conjecture (8) when $k \geq 3$ [Hei19]. The $k = 4$ case was entirely open since at least the 1990s [SM96, Problem 2] [Hut97]. The $k = 3$ case was partially resolved in [CCH+08] and then recently resolved in [MN22a]. The case of an arbitrary number $k \geq 3$ of sets with $t \to 0^+$ in (8) was then solved in [MN22a]. I then improved this result in [Hei19] by showing that sets close to minimizing the total Gaussian surface area are close to the optimal sets. The analogue of the $k = 3$ result [Hei21c, MN22a] for Lebesgue measure was resolved in a well-known 2002 result [HMRR02]. The analogue of the result of [MN22a] for Lebesgue measure with $k > 3$ sets was believed to be impossible, and perhaps it is now believed to be slightly less impossible.

The above results built upon my previous work [Hei14], where I showed the following. For any $n \geq 2$, there exists $t(n) > 0$ such that for any $t(n) < t < \infty$ and $k = 3$, the conjecture (8) holds. I used geometric and Fourier analytic arguments to show the first variation of (8) defines a contractive mapping, when restricted to partitions that almost achieve equality in (8). The result of [HT21] then makes the earlier result of [Hei14] dimension-independent.

In [HMN16], together with Mossel and Neeman, we showed that if the measure restriction of (8) is changed, then the most natural restatement of the conjecture (8) is false. Specifically, if $a_1, \ldots, a_k$ are real numbers with $0 < a_i < 1$ for all $i = 1, \ldots, k$ and $\sum_{i=1}^{k} a_i = 1$ with $(a_1, \ldots, a_k) \neq (1/k, \ldots, 1/k)$, and if $t > 0$, then the inequality (8) does not hold if we try to replace the sets $\{B_i\}_{i=1}^{k}$ with any set of simplicial cones that partition Euclidean space. This negative result implies that the endpoint case $t \to 0^+$ of (8) is quite different from the case $t > 0$, since the case $t \to 0^+$ holds for any measure restriction [MN22a] but the case $t > 0$ can only hold when all of the sets have equal Gaussian measures. Indeed, the proof of [MN22a] when $t \to 0^+$ proves their result by considering all possible measure restrictions simultaneously. My proof does not have this shortcoming [Hei21c] so that it might have a better chance of applying to (8).

2. A Sphere-Valued Borell Inequality

The MAX-CUT problem discussed in Section 1.5 has a matrix-valued analogue, known as Quantum MAX-CUT. As with MAX-CUT, it is natural to ask for approximation algorithms for Quantum MAX-CUT, and to try to prove sharp computational hardness of those algorithms [HNP+21, Kin22]. Below, we only discuss classical algorithms for Quantum MAX-CUT, i.e. we do not discuss quantum algorithms for Quantum MAX-CUT.

For the product state Quantum MAX-CUT problem, there is a conjecturally optimal approximation algorithm. Assuming the Unique Games Conjecture and Conjecture (11) below (a vector-valued Borell inequality), we would then have a sharp hardness of approximation for the product state of Quantum MAX-CUT. The vector-valued Borell inequality, Conjecture (11) [HNP+21] is a sphere-valued generalization of the Borell inequality [Bor85].

For any positive integer k, denote

$$S^{k-1} := \{x \in \mathbb{R}^k \colon \|x\| = 1\}$$

$$f_{\mathrm{opt}}(x) := \frac{x}{\|x\|}, \qquad \forall\, x \in \mathbb{R}^k \setminus \{0\}. \tag{10}$$

Let $n \geq k$ be positive integers. Let $f \colon \mathbb{R}^n \to S^{k-1}$ be measurable. Then the sphere-valued Borell inequality, as conjectured by [HNP+21] says the following: for any $n \geq k \geq 1$

$$\begin{aligned} \int_{\mathbb{R}^n} f d\gamma_n = 0 \quad \wedge \quad t > 0 \quad &\Longrightarrow \quad \int_{\mathbb{R}^n} \langle f, e^{-tL} f\rangle d\gamma_n \leq \int_{\mathbb{R}^k} \langle f_{\mathrm{opt}}, e^{-tL} f_{\mathrm{opt}}\rangle d\gamma_k \\ t > 0 \quad &\Longrightarrow \quad \int_{\mathbb{R}^n} \langle f, (-e^{-t})^L f\rangle d\gamma_n \geq \int_{\mathbb{R}^k} \langle f_{\mathrm{opt}}, (-e^{-t})^L f_{\mathrm{opt}}\rangle d\gamma_k. \end{aligned} \tag{11}$$

The case $n \geq k = 1$ of Conjecture (11) follows from Borell's inequality (4) and (5). (Borell's inequality can be rewritten as an inequality for functions $f\colon \mathbb{R}^n \to \{-1,1\} = S^0$.) In this way, Conjecture 11 generalizes Borell's inequality to the sphere-valued setting.

2.1. **Our Contribution.** In [Hei23a], we prove that Conjecture (11) holds for all $n \geq k = 3$ and for all $-.104 < \rho < .104$. (In fact, we prove a stronger "stable" version of Conjecture (11) when $n = k = 3$.)

Although we cannot prove all of Conjecture (11), our result [Hei23a] is sufficient to obtain the following corollary. Assume that the Unique Games Conjecture 6 is true. Then it is NP-hard to approximate the product state of Quantum MAX-CUT within a multiplicative factor of .9859. In contrast, a polynomial time algorithm is known for this problem with approximation factor .956....

To the author's knowledge, Theorem 9 is the only computational hardness result for the product state of Quantum MAX-CUT, besides the conjectural work of [HNP+21].

2.2. **Quantum MAX-CUT.** Below we describe the Quantum MAX-CUT problem by analogy with MAX-CUT. Computational hardness for Quantum MAX-CUT was the motivation for Conjecture (11). When M is a 2×2 matrix and j is a positive integer, we denote

$$M^{\otimes j} := \underbrace{M \otimes \cdots \otimes M}_{j \text{ times}}.$$

If n is a positive integer and $1 \leq j \leq n$, denote

$$Z_j := I_2^{\otimes(j-1)} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I_2^{\otimes(n-j)}, \qquad \forall\, 1 \leq j \leq n, \qquad I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The (weighted) **MAX-CUT** problem from Section 1.5 can be equivalently stated as [GP19, HNP+21]: given $w\colon \{1,\ldots,n\}^2 \to [0,\infty)$ satisfying $w_{ij} = w_{ji}$ and $w_{ii} = 0$ for all $1 \leq i,j \leq n$, compute the following quantity

$$\max_{u \in (\mathbb{C}^2)^{\otimes n}\colon \|u\| \leq 1} u^* \Big(\sum_{i,j=1}^{n} w_{ij} (I_2^{\otimes n} - Z_i Z_j) \Big) u.$$

Define now

$$X_j := I_2^{\otimes(j-1)} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I_2^{\otimes(n-j)}, \qquad \forall\, 1 \leq j \leq n,$$

$$Y_j := I_2^{\otimes(j-1)} \otimes \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \otimes I_2^{\otimes(n-j)}, \qquad \forall\, 1 \leq j \leq n.$$

The **Quantum MAX-CUT** problem is [GP19, Kin22, HNP+21]: given $w\colon \{1,\ldots,n\}^2 \to [0,\infty)$ satisfying $w_{ij} = w_{ji}$ and $w_{ii} = 0$ for all $1 \leq i,j \leq n$, compute the following quantity

$$\max_{u \in \mathbb{C}^{2n}\colon \|u\| \leq 1} u^* \Big(\sum_{i,j=1}^{n} w_{ij} (I_2^{\otimes n} - X_i X_j - Y_i Y_j - Z_i Z_j) \Big) u.$$

The **product state of Quantum MAX-CUT** is the more restricted optimization problem of computing

$$\max_{\substack{u = u_1 \otimes \cdots \otimes u_n\colon \\ u_i \in \mathbb{C}^2,\, \|u_i\| \leq 1,\, \forall\, 1 \leq i \leq n}} u^* \Big(\sum_{i,j=1}^{n} w_{ij} (I_2^{\otimes n} - X_i X_j - Y_i Y_j - Z_i Z_j) \Big) u.$$

3. Noninteractive Simulation of Correlated Random Variables

Let $m > 1$ be an integer. Let Z be a random variable with values in $\{1, \ldots, m\}$. Suppose we would like to simulate Z, e.g. on a computer. Let X be a standard Gaussian random variable, that we consider to be a source of randomness. It is a standard exercise to show that $\exists\, f\colon \mathbb{R} \to \{1, \ldots, m\}$ such that $f(X)$ is equal to Z in distribution, denoted as $f(X) \stackrel{d}{=} Z$. That is, any discrete distribution Z can be **simulated** from one sample of a Gaussian random variable X. However, determining whether or not correlated sources of randomness can simulate a given Z is a much harder problem.

Question 1 (**Noninteractive Simulation**, [SW73, GK73, Wit75, Wyn75])**.** *Let Z be a random variable with values in $\{1, \ldots, m\}^2$.*

Let X and Y be two real-valued random variables. Let $(X_1, Y_1), (X_2, Y_2), \ldots$ be independent identically distributed copies of (X, Y).

Do there exist $k \geq 1$, $f, g\colon \mathbb{R}^k \to \{1, \ldots, m\}$ such that

$$(f(X_1, \ldots, X_k),\; g(Y_1, \ldots, Y_k)) \stackrel{d}{=} Z?$$

More generally, for all $k \geq 1$, do there exist $f_k, g_k\colon \mathbb{R}^k \to \{1, \ldots, m\}$ such that

$$\lim_{k\to\infty} \sup_{A \subseteq \{1,\ldots,m\}^2} \left| \mathbb{P}\Big((f_k(X_1, \ldots, X_k),\; g_k(Y_1, \ldots, Y_k)) \in A\Big) - \mathbb{P}(Z \in A) \right| \stackrel{?}{=} 0?$$

If this last condition holds true, then we say that Z can be ***noninteractively simulated*** *by sampling from (X, Y).*

Generally speaking, Question 1 asks how much "correlation information" between two random variables can be transferred to another (pair of) random variables.

Question 1 can be phrased as a two-player game. Suppose there are two players A and B. Player A has access to $X_1, X_2, \ldots$ and player B has access to $Y_1, Y_2, \ldots$. Without communication, what joint probability distributions can players A and B jointly simulate?

Even for simple choices of X, Y, Z, very little can be said about Question (1). For example, let (X, Y) be uniform on the set of three points $\{(0, 0), (0, 1), (1, 0)\} \subseteq \mathbb{R}^2$. Then it is an open problem if a pair $(U, V) = Z$ of uniform $\{-1, 1\}$-valued random variables with correlation .49 can be noninteractively simulated by sampling from (X, Y) [KA16, GKR18].

Applications of the noninteractive simulation problem include: cryptography, design of error-correcting codes [MOR$^+$06, Yan07], and design of autonomous agents [KA16]. For example, an autonomous drone delivering a package might have to make decisions, using randomness, without consulting its dispatcher, due to a nonexistent cell-phone signal.

Some recent works [DMN17, DMN18, GKR18] considered Question (1) when $(X, Y) \in \mathbb{R}^2$ are ρ-correlated standard Gaussian random variables. (That is $\mathbb{E}X = \mathbb{E}Y = 0$, $\mathbb{E}X^2 = \mathbb{E}Y^2 = 1$ and $\mathbb{E}XY = \rho \in (-1, 1)$.) They showed that, for any $\varepsilon > 0$, one of the following two cases occurs in Question 1. Either:

- Z cannot be noninteractively simulated from (X, Y) with total variation error 10ε. That is, for all $k \geq 1$ and for all $f_k, g_k\colon \mathbb{R}^k \to \{1, \ldots, m\}$ the total variation distance of $(f_k(X_1, \ldots, X_k),\; g_k(Y_1, \ldots, Y_k))$ and Z is at most 10ε, or
- There exists $k \geq 1$ and $f_k, g_k\colon \mathbb{R}^k \to \{1, \ldots, m\}$ such that the total variation distance of $(f_k(X_1, \ldots, X_k),\; g_k(Y_1, \ldots, Y_k))$ and Z is at most ε.

However, the k that is achieved in the second case of the above dichotomy is exponential in m, ε^{-1} and $(1 - |\rho|)^{-1}$ [GKR18, Theorem 1.1]. This result was an improvement over previous non-explicit bounds [DMN17, DMN18]. So, after [GKR18], it was still not clear whether or not the k appearing in the second case above could be finite or not. We show in fact that it can be made finite, i.e. it only depends quadratically in m and it does not depend on ε or ρ.

3.1. **Our Contribution.** When X and Y are standard Gaussians with fixed correlation $\rho \in (-1,1)$, we show that the set of probability distributions that can be noninteractively simulated from k Gaussian samples is the same for any $k \geq m^2$. Previously, it was not even known if this number of samples m^2 would be finite or not, except when $m \leq 2$. That is [HT22], we showed the following holds, for any Z in Question 1: either

- Z cannot be noninteractively simulated from (X,Y), or
- There exist $f,g\colon \mathbb{R}^{m^2} \to \{1,\ldots,m\}$ such that

$$(f(X_1,\ldots,X_{m^2}),\ g(Y_1,\ldots,Y_{m^2})) \stackrel{d}{=} Z.$$

That is, m^2 samples suffice for noninteractive simulation from Gaussian sources.

The proof uses a modification of the variational methods used in [Hei21e].

3.2. **Further Directions.** Due to Central Limit Theorem types of arguments [DMN17, DMN18, GKR18], our main result gives an algorithm for deciding whether or not a random variable Z can be nonineractively simulated from *arbitrary* (i.e. potentially non-Gaussian) inputs (X,Y) in Question 1. However, the algorithm has a triple exponential run time, and our bounds only marginally improve on those bounds already given in [GKR18].

The following questions remain open from our work [HT22].

- Which probability distributions can be noninteractively simulated from Gaussians? That is, is there some "easy" description of these probability distributions? (Currently, it is only possible to give a precise answer when $m \leq 2$.) Our result does not explicitly describe these probability distributions.
- There is a brute-force search algorithm associated with our main result [HT22] for deciding whether or not (U,V) is ε-close to being noninteractively simulated from correlated Gaussians (X,Y). (The algorithm traverses an appropriate ε-net over a set of polynomials f,g of bounded degree.) This algorithm has an exponential run time in $1/\varepsilon, m$ and $\log|\rho|$. Is there something more efficient than this brute-force search? That is, can the run time of this algorithm be decreased substantially?
- Same as the previous question, but for general inputs. As discussed in [GKR18], the run time of the more general algorithm is triple exponential in all parameters. But is something better possible? That is, can the run time of this algorithm be decreased substantially?

I am currently working on extending the result of [HT22] to the case of multiparty noninteractive simulation. For example, we have the following "three-party" version of Question 1.

Question 2 (**Three Party Noninteractive Simulation**, [GKS16])**.** *Let Z be a random variable with values in $\{1,\ldots,m\}^2$.*

Let W,X and Y be three real-valued random variables. Let $(W_1,X_1,Y_1),(W_2,X_2,Y_2),\ldots$ be independent identically distributed copies of (W,X,Y).

Do there exist $k \geq 1$, $f,g,h\colon \mathbb{R}^k \to \{1,\ldots,m\}$ such that

$$(f(W_1,\ldots,W_k),\ g(X_1,\ldots,X_k),\ h(Y_1,\ldots,Y_k)) \stackrel{d}{=} Z?$$

More generally, for all $k \geq 1$, do there exist $f_k,g_k,h_k\colon \mathbb{R}^k \to \{1,\ldots,m\}$ such that

$$\lim_{k\to\infty} \sup_{A\subseteq\{1,\ldots,m\}^2} \Big|\mathbb{P}\Big((f_k(W_1,\ldots,W_k),\ g_k(X_1,\ldots,X_k),\ h_k(Y_1,\ldots,Y_k)) \in A\Big) - \mathbb{P}(Z\in A)\Big| \stackrel{?}{=} 0?$$

It seems that the methods of [HT22] are general enough to apply to Question 2 as well, and to more general multiparty noninteractive simulation.

4. Barthe's Symmetric Gaussian Problem

Borell's inequality (4) [Bor85] implies (3): half spaces have the smallest Gaussian surface area among all Euclidean subsets of fixed Gaussian volume. Half spaces lie on one side of a hyerplane, so these sets are not symmetric with respect to reflection across the origin. If we try to instead minimize Gaussian surface area among *symmetric* sets of fixed Gaussian volume, it is a priori unclear what set is the best, since half spaces are now excluded. For this reason, Barthe [Bar01] posed Conjecture 3 below in 2001. I also planned to consider Conjecture 3 as a test case for developing methods for related questions with more applications such as (8) above. This plan succeeded, as the methods developed in [Hei21b, Hei21d] were then applied in my later work [Hei21c].

4.1. Barthe's Symmetric Gaussian Problem.

Conjecture 3 ([Bar01])**.** *Let $A \subseteq \mathbb{R}^n$ have the smallest Gaussian surface area $\gamma_{n-1}(\partial A)$ among all sets of fixed Gaussian volume $\gamma_n(A)$, subject to the constraint $A = -A$. Then ∂A must be a* ***round cylinder****. That is, after applying a rotation to ∂A, $\exists\, r > 0$ and $\exists\, 0 \leq k \leq n-1$ such that $\partial A = rS^k \times \mathbb{R}^{n-k-1}$, where $S^k = \{(x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} \colon x_1^2 + \cdots + x_{k+1}^2 = 1\}$.*

If we remove the constraint $A = -A$ from Conjecture 3, then this problem is well understood by (3). However, all known proofs of (3) (with the exception of [MR15, BBJ17]) seem unable to handle the additional constraint that the set A is symmetric (i.e., that $A = -A$). In the work [Hei21d], I demonstrate that the calculus of variations techniques of [CM12, MR15, BBJ17] succeed in this task, where other proof strategies seem insufficient.

4.2. Our Contribution: Gaussian Surface Area of Symmetric Sets.

Let $A \subseteq \mathbb{R}^n$ have minimal Gaussian surface area among all sets satisfying $A = -A$ with fixed Gaussian measure $\gamma_n(A)$. A standard first variation argument shows that there exists $\lambda \in \mathbb{R}$ such that $H(x) = \langle x, N(x)\rangle + \lambda$ for all $x \in \partial A$, where $N(x)$ denotes the exterior pointing unit normal vector at x, and $H(x)$ denotes the mean curvature at x (i.e. the divergence of $N(x)$).

Below we say that a set $A \subseteq \mathbb{R}^n$ is a cylinder if A is a rotation of a set of the form $A' \times \mathbb{R}$ for some $A' \subseteq \mathbb{R}^{n-1}$.

We now state our main result in [Hei22a]: if $A \subseteq \mathbb{R}^n$ is a cylinder, then either A or A^c must be convex. Moreover, Conjecture 3 holds true, except possibly when H and λ have different signs. And when H and λ have the same sign, we show that Conjecture 3 holds with $\sqrt{n-1} \leq r \leq \sqrt{n+1}$ when $k \geq 1$.

It is expected in Conjecture 3 that, when n is large, the symmetric set minimizing Gaussian surface area is a cylinder, except when $\gamma_n(A) = 1/2$. So, the assumption of [Hei22a] that A be a cylinder is quite mild.

The main tool of [Hei21d, Hei19, Hei22a] is the Colding-Minicozzi theory for Gaussian minimal surfaces described in Section 3. A key new ingredient is the use of a randomly chosen degree 2 polynomial in the second variation (i.e., second derivative) formula for the Gaussian surface area.

Using a fairly different "penalty function" approach as compared to my earlier results such as [Hei21d], Barchiesi et al. [BJ20] proved Conjecture 3 when the measure restriction is close to one (or zero). It is unclear if their method can deal with other measure constraints, as in [Hei22a].

4.3. Potential Developments.

There are examples of curves with $n = 2$ and non-cylindrical surfaces with $n > 2$ where $H(x) = \langle x, N(x)\rangle + \lambda$ for all $x \in \partial A$, and where $H > 0$ and $\lambda < 0$ [Cha17, CLW22]. In particular, when $n = 2$, I currently do not know a general way to rule out these curves from Conjecture 3, but at least the remaining challenge is clear.

In the near future, I hope to extend the results of [Hei21d, Hei22a] to noise stability. That is, we want to prove the same result as [Hei21d], but with noise stability (as in (4)) replacing the Gaussian surface area. This problem, Conjecture 4 below, was first mentioned in [CR11, O'D12] in relation to the Gap-Hamming-Distance problem from communication complexity. The methods of

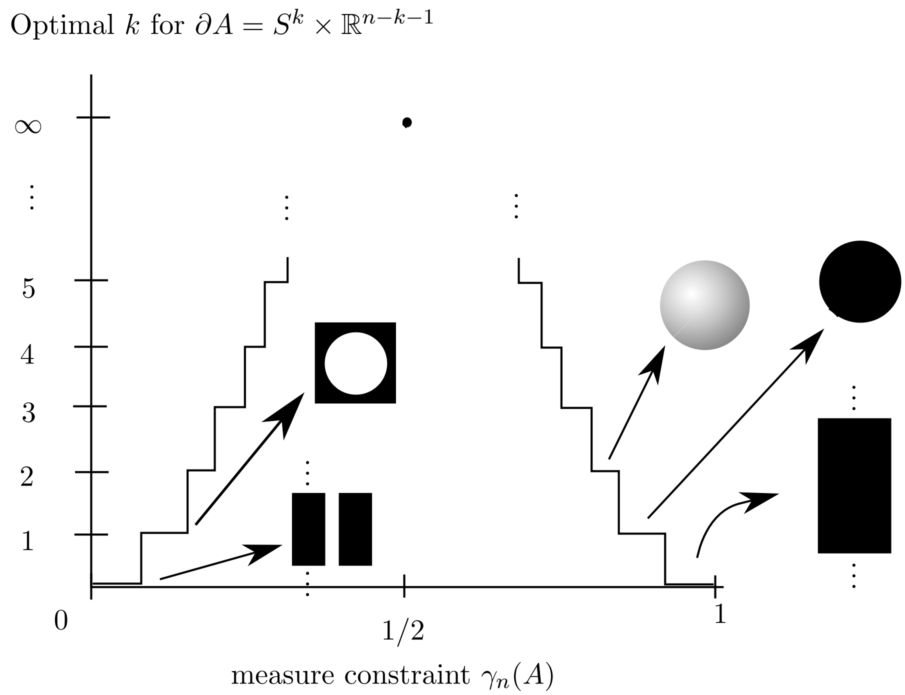

FIGURE 2. Frank Morgan has conjectured that the symmetric set of smallest Gaussian surface area has boundary $S^k \times \mathbb{R}^{n-k-1}$, where k is a function of the measure constraint as depicted here. So, Morgan's Conjecture refines Conjecture 3 above.

[Hei21d, Hei22a] are fairly different from the earlier work [Hei21b], where we computed the second variation of noise stability of the ball and its complement. The newer result [Hei22a] relies on using various functions of the curvature of the boundary within the second variation formula itself. It seems difficult to adapt this argument to the setting of noise stability, since the curvature of a set does not fit so naturally into the second variation formula for noise stability. So, for the following noise stability problem, it remains a challenge to find a suitable substitute for the curvature of a set that "fits nicely" into the second variation formula for noise stability.

Conjecture 4 (**Symmetric Gaussian Problem**, A Generalization of Conjecture 3)**.** *Let $t > 0$. Let $A \subseteq \mathbb{R}^n$ have the largest noise stability $\int_{\mathbb{R}^n} 1_A \cdot e^{-tL} 1_A d\gamma_n$ among all sets of fixed Gaussian volume $\gamma_n(A)$, subject to the constraint $A = -A$. Then ∂A must be a round cylinder.*

Below, we propose methods for extending the results of [Hei21d, Hei22a] to other related problems discussed in Sections 1 and 7.

4.4. **Our Contribution: Local Version of Conjecture 4 for large** t**.** In an earlier work [Hei21b], I show that Conjecture 4 is true for $n = 1$ and t sufficiently large, by simplifying the argument from my earlier work on the Standard Simplex Conjecture [Hei14] mentioned in Section 1. When $n \geq 2$, I compute the second variation of the ball and its complement, adapting a second variation argument from [CS07]. It turns out that the computation of the second variation of noise stability of the ball essentially reduces to proving an L_2 Poincaré inequality on the sphere.

5. Robust Borell Inequality

Borell's inequality (4), says that a Euclidean set of fixed Gaussian volume with largest noise stability must be a half space.

A robust version of Borell's above inequality says: if a Euclidean set A nearly maximizes noise stability $\int_{\mathbb{R}^n} 1_A \cdot e^{-tL} 1_A d\gamma_n$ (subject to the Gaussian volume $\gamma_n(A)$ being fixed), then the set A is close to a half space. Robust versions of Borell's inequality were proven in [MN15b, Eld15].

The proof of the related robust Gaussian isoperimetric inequality in [BBJ17] uses the calculus of variations to minimize the Gaussian surface area plus a "penalty" function. The minimum of this quantity occurs at a half space, so that the "penalty" function quantifies how far an arbitrary set is from being a half space. The main step of the proof computes the second derivative of infinitesimal translations of an optimal set that are Gaussian volume-preserving.

The proof methods of the more general robust Borell inequality [MN15b, Eld15] are arguably ad hoc, so one might hope for a more elementary proof, along the lines of [BBJ17]. Moreover, the proof methods of [MN15b, Eld15] do not seem to generalize to inequalities for the noise stability of partitions of Euclidean space, as opposed to the calculus of variations arguments of e.g. [Hei19, HT21].

5.1. **Our Contribution.** In the paper [Hei21e], I demonstrate that the penalty function method of [BBJ17] can prove a robust Borell inequality. Moreover, I prove some cases of a conjecture of Eldan [Eld15], thereby giving a variational proof of a robust version of Borell's inequality. The main result of [Hei21e] combined with previous works such as [BBJ17, Hei19, HT21] essentially shows that one single argument can prove nearly every known inequality for sets or partitions that maximize noise stability, with respect to Gaussian volume constraints. So, there is now essentially a single proof method that yields all known inequalities for noise stability of Euclidean sets and partitions.

As shown in [MN15b], robust Borell inequalities imply robust majority is stablest theorems, thereby giving robust versions of Arrow's Theorem in social choice theory. Also, a robust Borell inequality implies that any nearly optimal algorithm for the MAX-CUT problem must be "close" to the random hyperplane semidefinite programming algorithm of Goemans-Williamson.

6. The Unique Games Conjecture and Gaussian Isoperimetry

6.1. **The Unique Games Conjecture.** The Unique Games Conjecture [Kho02] is a standard assumption in theoretical computer science. This conjecture can be considered a contemporary proxy for the assumption that P≠NP. That is, proving or disproving the Unique Games Conjecture is expected to have similar significance and consequences to proving or disproving P≠NP. Moreover, both problems are closely related. As we will describe below, the Unique Games Conjecture can be succinctly stated as: approximate linear algebra is hard.

A recent breakthrough of [KMS18] gives significant positive evidence for The Unique Games Conjecture 6. In a certain sense, the result [KMS18] proves Conjecture 6 "half way." The remaining open cases of the Unique Games Conjecture 6 are closely related to Conjecture 8 discussed in the following section [Hei20b].

Definition 5 (Γ-MAX-2LIN(p)). Let $p \geq 2$ be a prime number. We define the Γ-MAX-2LIN(p) problem. In this problem, we are given $n \in \mathbb{N}$ and $2n$ variables $x_i \in \mathbb{Z}/p\mathbb{Z}$, $i \in \{1, \ldots, 2n\}$. We are also given a matrix $\{a_{ij}\}_{i,j=1}^{2n}$ with $a_{ij} \geq 0$ for all $i, j \in \{1, \ldots, 2n\}$ and a set $E \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ with n elements. An element $(i, j) \in E$ corresponds to one of n linear equations of the form $x_i - x_j = c_{ij} (\bmod\, p)$, where $c_{ij} \in \mathbb{Z}/p\mathbb{Z}$. The goal of the Γ-MAX-2LIN(p) problem is to find the following quantity:

$$\max_{(x_1,\ldots,x_{2n})\in(\mathbb{Z}/p\mathbb{Z})^{2n}} \sum_{\substack{(i,j)\in E:\\ x_i-x_j=c_{ij}(\bmod\, p)}} a_{ij}. \tag{12}$$

That is, we need to maximize the (weighted) number of equations $x_i - x_j = c_{ij} (\bmod\, p)$ that are satisfied.

Conjecture 6 (**Unique Games Conjecture**, [Kho02, KKMO07])**.** *For every $\varepsilon \in (0,1)$, there exists a prime number $p(\varepsilon)$ such that no polynomial time algorithm (with respect to the parameter n) can distinguish between the following two cases, for instances of Γ-MAX-2LIN($p(\varepsilon)$) with $a_{ij} = 1$ for all $i,j \in \{1,\ldots,2n\}$:*

(i) (12) *is larger than $(1-\varepsilon)n$, or*
(ii) (12) *is smaller than εn.*

If (12) were equal to n, then we could find $(x_1,\ldots,x_{2n})$ achieving the maximum in (12) by Gaussian elimination. One can therefore interpret the Unique Games Conjecture as an assertion that approximate linear algebra is hard. The truth or falsity of this conjecture remains a major open problem.

6.2. **Khot and Moshkovitz's new approach.** Recently, [KM16] showed that a special case of the Unique Games Conjecture follows from the following conjecture. Recall that $v \in \mathbb{R}^n$ is a standard basis vector if one of its coordinates is 1 and all of its other coordinates are zero. Let $A \subseteq \mathbb{R}^n$ with $-A = A^c$ and such that $A + v = A^c$ for every standard basis vector $v \in \mathbb{R}^n$. That is, let A be a "periodic set." Then the noise stability of A is at most the noise stability of the set

$$H := \Big\{x = (x_1,\ldots,x_n) \in \mathbb{R}^n \colon \sin(\pi \sum_{i=1}^n x_i) \geq 0\Big\}.$$

Note that the boundary ∂H of H is the set of parallel hyperplanes of the form

$$\{x = (x_1,\ldots,x_n) \in \mathbb{R}^n \colon \exists\, k \in \mathbb{Z} \text{ such that } x_1 + \cdots + x_n = k\}.$$

For this reason, H is called a "periodic half space."

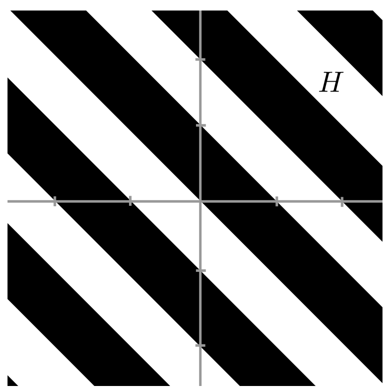

FIGURE 3. A periodic half space H.

Conjecture 7 ([KM16])**.** *Let $A \subseteq \mathbb{R}^n$ with $-A = A^c$ and such that $A + v = A^c$ for every standard basis vector $v \in \mathbb{R}^n$. Then for any $0 < t < 1$,*

$$\int_{\mathbb{R}^n} 1_A \cdot e^{-tL} 1_A d\gamma_n \leq \int_{\mathbb{R}^n} 1_H \cdot e^{-tL} 1_H d\gamma_n.$$

The case $t \to 0^+$ of Conjecture 7 is most relevant for the implications for the Unique Games Conjecture in [KM16]. So, as a first step, it is sensible to try to prove Conjecture 7 for the Gaussian surface area functional. That is, as a first step, we instead prove the following special case of Conjecture 7.

Conjecture 8 (Endpoint Case of Conjecture 7)**.** *Let $A \subseteq \mathbb{R}^n$ with $-A = A^c$ and such that $A + v = A^c$ for every standard basis vector $v \in \mathbb{R}^n$. Then*

$$\gamma_{n-1}(\partial A) \geq \gamma_{n-1}(\partial H).$$

6.3. **Our Contribution.** In [Hei20b], we prove Conjecture 8, less a small error.

Theorem 9 ([Hei20b])**.** *Let $A \subseteq \mathbb{R}^n$ with $-A = A^c$ and such that $A + v = A^c$ for every standard basis vector $v \in \mathbb{R}^n$. Then*
$$\gamma_{n-1}(\partial A) \geq (1 - 6 \cdot 10^{-9})\gamma_{n-1}(\partial H).$$

Actually, we also incorporate a "robustness" term that says if ∂A is far from ∂H, then $\gamma_{n-1}(\partial A)$ is much larger than $\gamma_{n-1}(\partial H)$. That is, we show in [Hei20b] that
$$\gamma_{n-1}(\partial A) \geq (1 - 6 \cdot 10^{-9})\gamma_{n-1}(\partial H) + \int_{\partial A} \Big(1 - \frac{\|N(x)\|_1}{\sqrt{n}}\Big)\gamma_n(x)dx.$$
Here $N(x)$ is a unit normal vector at $x \in \partial A$.

The proof is fairly elementary. Using the Poisson Summation Formula, Conjecture 8 can be rewritten as an isoperimetric problem on the torus equipped with the heat kernel measure. This heat kernel measure is very close to being constant. So, a Euclidean projection argument can then prove Theorem 9.

Using standard arguments, these results imply the following weak version of Conjecture 7.

Theorem 10 (Weak Version of Conjecture 7, [Hei20b])**.** *Let $A \subseteq \mathbb{R}^n$ with $-A = A^c$ and such that $A + v = A^c$ for every standard basis vector $v \in \mathbb{R}^n$. Then for any $0 < t < 1/2$,*
$$\int_{\mathbb{R}^n} 1_A \cdot e^{-tL} 1_A d\gamma_n \leq \int_{\mathbb{R}^n} 1_H \cdot e^{-tL} 1_H d\gamma_n + 3 \cdot 10^{-9} - \frac{\sqrt{1 - e^{-2t}}}{\sqrt{2\pi}} \int_{\partial\Omega} \Big(1 - \frac{\|N(x)\|_1}{\sqrt{n}}\Big)\gamma_n(x)dx + o(t).$$

Here the implied constant can depend on A. Note that if t is near zero then this bound is typically vacuous.

In summary, our results give indirect evidence for the truth of the Unique Games Conjecture, Conjecture 6.

6.4. **Potential Developments.** It is tempting to try to apply the variational methods of [BBJ17, Hei21d, Hei21c] to Conjecture 7. However, the error terms seem difficult to control, since many oscillatory terms appear. For example, if we try to use the Colding-Minicozzi theory from Section 1.3, we then must try to find the eigenfunctions of the operator
$$\Delta_\Sigma + \Big\langle \frac{-\nabla \sum_{z \in \mathbb{Z}^n} \gamma_n(x + z)}{\sum_{z \in \mathbb{Z}^n} \gamma_n(x + z)}, \nabla_\Sigma \Big\rangle - \|F\|_2^2 - 1, \qquad \forall\, x \in \Sigma = \partial A.$$
However, it seems difficult to identify or even approximate these eigenfunctions.

Even if the Unique Games Conjecture is eventually proven false, Conjectures 7 and 8 will still be of interest. Some similar isoperimetric problems on the torus are motivated by the physics of ferromagnetic materials, and these problems are predicted to have optimal sets consisting of parallel stripes, as in Conjectures 7 and 8. This "crystallization" prediction has been studied in e.g. [The06, BPT13, GM12, GS16, DR18], though these studies have typically focused only on $n = 2$ or $n = 3$, whereas Conjectures 7 and 8 are stated for any dimension $n \geq 1$. The isoperimetric problem on the flat torus equipped with the Lebesgue measure also appears to be unsolved [Ros01], so no methods from that problem seem relevant to Conjecture 8. Also, the problems studied in [The06, BPT13, GM12, GS16, DR18] still appear to be unsolved, so progress on Conjectures 7 and 8 could lead to progress on these other problems.

7. The Propeller Conjecture

7.1. **Invariance Principle.** The invariance principles of [Rot79, Cha06, MOO10] are nonlinear versions of the Central Limit Theorem, with error bounds. That is, the invariance principle implies the Berry-Esséen Central Limit Theorem, which we now recall.

Let n be a positive integer. Let $x_1,\ldots,x_n$ be commutative indeterminate variables, and let

$$Q(x_1,\ldots,x_n) := \frac{x_1+\cdots+x_n}{\sqrt{n}}.$$

Let $b_1,\ldots,b_n$ be independent identically distributed (i.i.d.) uniform random variables in $\{-1,1\}$, and let $g_1,\ldots,g_n$ be i.i.d. standard Gaussian random variables. Letting $\mathbb{E}$ denote expected value, we then define the 2-norm of Q to be

$$\|Q\|_2 := (\mathbb{E}\,|Q(b_1,\ldots,b_n)|^2)^{1/2}. \tag{13}$$

The **Berry-Esséen Central Limit Theorem** then says

$$\sup_{t\in\mathbb{R}} |\mathbb{P}(Q(b_1,\ldots,b_n)\leq t) - \mathbb{P}(Q(g_1,\ldots,g_n)\leq t)| \leq 3 \max_{i=1,\ldots,n}\left\|\frac{\partial}{\partial x_i}Q\right\|_2. \tag{14}$$

If the rightmost expression looks unfamiliar, note that $Q(g_1,\ldots,g_n)$ has a standard Gaussian distribution, and $\|\frac{\partial}{\partial x_i}Q\|_2 = 1/\sqrt{n}$ for all $i\in\{1,\ldots,n\}$. The proof of (14) can also be extended to moments of Q:

$$\left|\mathbb{E}\,|Q(b_1,\ldots,b_n)|^4 - \mathbb{E}\,|Q(g_1,\ldots,g_n)|^4\right| \leq 240 \max_{i=1,\ldots,n}\left\|\frac{\partial}{\partial x_i}Q\right\|_2. \tag{15}$$

A similar statement can be made for higher moments of Q. The commutative invariance principle [Rot79, Cha06, MOO10] implies, among other things, that (15) holds for multilinear polynomials.

Let $d\in\mathbb{N}$. Let $Q(x_1,\ldots,x_n)$ be a multilinear polynomial of degree d, so that

$$Q(x_1,\ldots,x_n) = \sum_{S\subseteq\{1,\ldots,n\}\colon\, |S|\leq d} c_S\prod_{i\in S}x_i,\quad c_S\in\mathbb{R},\,\forall\, S\subseteq\{1,\ldots,n\}.$$

Assume that $\|Q\|_2\leq 1$. Then the **commutative invariance principle** [MOO10] says that

$$\left|\mathbb{E}\,|Q(b_1,\ldots,b_n)|^4 - \mathbb{E}\,|Q(g_1,\ldots,g_n)|^4\right| \leq 24\cdot 10^d \max_{i=1,\ldots,n}\left\|\frac{\partial}{\partial x_i}Q\right\|_2. \tag{16}$$

The commutative invariance principle (16) in [MOO10] is proven by a combination of the Lindeberg replacement argument and the hypercontractive inequality [Sta59, Fed69, Bon70, Nel73, Gro75, Bec75] (see (21) below). That is, one replaces one argument of Q at a time, adding up the resulting errors and controlling them via the hypercontractive inequality. The invariance principle (16) has seen many applications [O'D, O'D14] in recent years. Here is a small sample of such applications and references: isoperimetric problems in Gaussian space and in the hypercube [MOO10, IM12], social choice theory, Unique Games hardness results [KKMO07, IM12], analysis of algorithms [BR15], random matrix theory [MP14], free probability [NPR10], optimization of noise sensitivity [Kan14], etc.

7.2. **Grothendieck's Inequality.** The first application of (16) we will describe is computational hardness for the commutative Grothendieck inequality [RS09]. For any $x=(x_1,\ldots,x_n), y=(y_1,\ldots,y_n)\in\mathbb{C}^n$, define $\langle x,y\rangle := \sum_{i=1}^n x_i\overline{y_i}$ and define $\|x\|_2 := \sqrt{\langle x,x\rangle}$. Let $\{a_{ij}\}_{i,j=1}^n$ be a real matrix. Proven first in 1953, **Grothendieck's Inequality** [Gro53, LP68, AN06, BMMN13] says there exists a constant $K>0$ which does not depend on n or on $\{a_{ij}\}_{i,j=1}^n$ such that

$$\sup_{\substack{w_1,\ldots,w_n,r_1,\ldots,r_n\in\mathbb{R}^{2n-1}\\ \|w_i\|_2=\|r_i\|_2=1,\,\forall\, i=1,\ldots,n}} \sum_{i,j=1}^n a_{ij}\langle w_i,r_j\rangle \leq K\cdot \sup_{u_1,\ldots,u_n,\nu_1,\ldots,\nu_n\in\{-1,1\}}\sum_{i,j=1}^n a_{ij}u_i\nu_j. \tag{17}$$

That is, for a general optimization problem (corresponding to the left side of (17)), it is possible to "round" the unit vectors w_i,r_i, $i=1,\ldots,n$ to a one-dimensional set of unit vectors u_i,ν_i, $i=1,\ldots,n$. And the weighted sum of inner products of the vectors does not decrease very much after we perform this rounding procedure. It is known that $K<\pi/(2\log(1+\sqrt{2}))$ [BMMN13], and that a rounding procedure can establish the best constant in Grothendieck's inequality [NR14]. However, it remains a major open problem to find this optimal rounding procedure and to find the smallest possible constant K in Grothendieck's inequality.

Finding the smallest possible constant K in (17) has several interpretations beyond mathematics. From the physics perspective, the best constant K in (17) is also the smallest constant in certain Bell inequalities in quantum mechanics [Pis12]. More specifically, Bell's inequality says that the smallest K possible in (17) satisfies $K > 1$. From the computer science perspective, assuming the Unique Games Conjecture (see Section 6.1), it is impossible to approximate the right side of (17), in time polynomial in n, within a multiplicative factor smaller than K, where K is the smallest possible constant in the inequality (17) [RS09]. Mathematically, (17) can be rewritten as a ratio between two tensor product norms. One could consider this breadth of interpretation as evidence for the difficulty and importance of finding the smallest possible value of K in (17).

In an effort to better understand Grothendieck's inequality (17), and to approximate the optima of kernel clustering problems from machine learning, Khot and Naor [KN09, KN13] investigated Grothendieck's inequality (17) for $\{a_{ij}\}_{i,j=1}^{n}$ that are positive semidefinite.

7.3. **Generalized Positive Semidefinite Grothendieck Inequalities.** Suppose $\{a_{ij}\}_{i,j=1}^{n}$ is a positive semidefinite matrix, i.e. a real symmetric matrix with all eigenvalues nonnegative. Let $\nu_1, \ldots, \nu_k \in \mathbb{R}^k$ with $k \geq 2$, and let $B = \{b_{ij}\}_{i,j=1}^{k}$ be the symmetric positive semidefinite matrix with $b_{ij} := \langle \nu_i, \nu_j \rangle$. Then the **Generalized Positive Semidefinite Grothendieck Inequality** [KN13, Theorem 3.1, Theorem 3.3] says that there exists $C(B) > 0$ which does not depend on n or on $\{a_{ij}\}_{i,j=1}^{n}$ such that

$$\max_{\substack{w_1,\ldots,w_n \in \ell_2 \\ \|w_i\|_2 = 1,\, \forall\, i=1,\ldots,n}} \sum_{i,j=1}^{n} a_{ij} \langle w_i, w_j \rangle \leq \frac{1}{C(B)} \cdot \max_{\sigma \colon \{1,2,\ldots,n\} \to \{1,2,\ldots,k\}} \sum_{i,j=1}^{n} a_{ij} \langle \nu_{\sigma(i)}, \nu_{\sigma(j)} \rangle. \tag{18}$$

Instead of rounding the vectors w_i, $i = 1, \ldots, n$ to 1 or -1 as in Grothendieck's inequality (17), Khot and Naor prove (18) by rounding the vectors w_i, $i = 1, \ldots, n$ to k vectors ν_i, $i = 1, \ldots, k$. Also, the best constant $1/C(B)$ is found by finding the best rounding procedure.

7.4. **The Sharp Constant of Grothendieck Inequalities.** Let I_k denote the $k \times k$ identity matrix. In [KN09, KN13], it is shown that $C(B)$ can be found by solving a finite dimensional optimization problem, whose parameters depend on B. However, this optimization problem is non-convex in general, so standard methods cannot compute $C(B)$. The **Propeller Conjecture** guesses the value of $C(B)$ for $B = I_k$, $k \geq 4$:

$$C(I_k) := \sup_{\substack{A_1,\ldots,A_k \colon \cup_{i=1}^{k} A_i = \mathbb{R}^{k-1}, \\ \gamma_{k-1}(A_i \cap A_j) = 0,\, \forall\, i,j \in \{1,\ldots,k\},\, i \neq j}} \sum_{i=1}^{k} \left\| \int_{A_i} x \, d\gamma_{k-1}(x) \right\|_2^2 = \frac{9}{8\pi} = C(I_3). \tag{19}$$

The constant $C(I_3) = 9/(8\pi)$ is computed in [KN09] using Lagrange multipliers, but $C(I_4)$ does appear to be computable using this technique.

7.5. **Our Contribution.** Together with Naor and Jagannath, using some theoretical results and a brute force search, we give a computer-assisted proof of the Conjecture (19) in the case $k = 4$ [HJN13]. The analytic results use a connection between the maximization problem (19) and discrete harmonic maps into the sphere. Intuition derived from this connection allows several ad hoc arguments to rule out candidates for partitions that maximize (19).

7.6. **Potential Developments.** Solving (19) for all $k \geq 4$ would yield better understanding of semidefinite programming algorithms and potential insight into computing the best constant in Grothendieck's inequality (17).

8. Independent Sets in Random Graphs and Random Trees

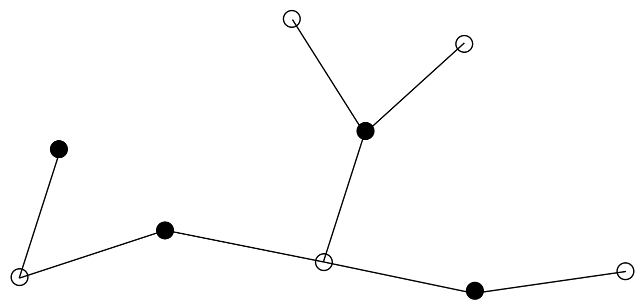

Figure 4. The open circles form an independent set of size 5 in the tree. The filled in circles form an independent set of size 4 in the tree.

Let G be a finite, undirected graph with no self-loops and no multiple edges, on $n \geq 1$ labelled vertices $V := \{1, \ldots, n\}$, with edges $E \subseteq \{\{i, j\} : i, j \in V, i \neq j\}$. Let $0 \leq k \leq n$. An **independent set** of size k in G is a subset of vertices no two of which are connected by an edge. Let $x_k = x_k(G)$ denote the number of independent sets in size k in G. (Note that $x_0(G) = 1$ since we consider the empty set to be a subset of $V = \{1, \ldots, n\}$.) We refer to the sequence $x_0(G), \ldots, x_n(G)$ as the **independent set sequence** of G.

Question 11. *What does the sequences $x_0, x_1, \ldots, x_n$ "look like" for random graphs?*

Some motivations for this question include:

- Statistical physics, where an independent set represents molecules in a magnet that do not want to be close to each other;
- Computer Science and Probability, where many combinatorial optimization problems (k-SAT, MAX-Independent Set, etc.) exhibit similar interesting behavior for random instances. See for example the phase transition known as "shattering" discussed in e.g. [CE15] and [DSS16].
- Combinatorics, where one would like to exactly or approximately find the numbers $x_0, \ldots, x_n$ for both deterministic and random graphs.

We note that a classic NP-complete problem is: For any $k \geq 1$ and any graph G, decide whether or not $x_k(G) > 0$. So, it could be hard for a computer to decide whether or not a large graph has a large independent set. Counting the *number* of independent sets of a given size is then computationally more difficult. In fact, a remarkable result of [JSV04] implies a computational equivalence between approximately counting combinatorial quantities (such as independent sets), and constructing a stochastic process whose distribution converges to the uniform distribution on those combinatorial objects.

Intuitively, $x_0, \ldots, x_n$ should resemble the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$. Note that if $E = \emptyset$, $x_k = \binom{n}{k}$ for all $0 \leq k \leq n$. The sequence of binomial coefficients is unimodal, so one might expect the independent set sequence of a random graph to have this same behavior.

8.1. Independent Sets of Trees.

A **tree** on n vertices is a connected graph with no cycles. The following precise form of the rather vague Question 11 for trees was stated by Alavi, Malde, Schwenk and Erdős in 1987.

Question 12 ([AMSE87])**.** *Does every tree have a unimodal independent set sequence?*

Despite much effort, including [LM02, LM03, Zhu07, WZ11, Gal11, Gal12, BBO14, Zhu16, GH18], Question 12 remains open. The cited works mostly focus on answering Question 12 for particular families of trees. One general partial result towards Question 12 is the following.

Theorem 13 ([LM07]). *Let T be a tree whose largest independent set is of size j. Then the "last third" of the independence set sequence is unimodal:*

$$x_{\lceil(2j-1)/3\rceil}(T) \geq x_{1+\lceil(2j-1)/3\rceil}(T) \geq \cdots \geq x_{j-1}(T) \geq x_j(T).$$

That is, Question 12 is "one-third true."

Motivated by a question of Galvin, instead of trying to answer Question 12 for deterministic families of trees, we attempt to answer question two for random trees, with high probability.

A **random tree** T on $n \geq 2$ vertices is a random graph that is equal to any of the n^{n-2} possible labelled trees on n vertices, each with probability $1/n^{n-2}$.

8.2. **Our Contribution.**

Theorem 14 (**Partial Unimodality for Random Trees**, [Hei20a]). *There exists $c > 0$ such that, with probability at least $1 - e^{-cn}$, a random tree T on n vertices satisfies*

$$x_0(T) < x_1(T) < \cdots < x_{\lfloor(.26543)n\rfloor}(T).$$

For comparison, the largest independent set size in a random tree is about $.567143n$, with fluctuations of order $\sqrt{n}$, by the Azuma-Hoeffding inequality [Fri90]. So, the first 46.8% of the nontrivial independent set sequence is unimodal. Combined with Theorem 13 of Levit and Mandrescu, Question 12 is "four-fifths true", with high probability.

The proof of Theorem 14 required proving a concentration inequality for the number of neighbors in a random tree conditioned on having an independent set of fixed size [Hei22d]. The proof of this concentration inequality used a bijection between random trees and random mappings (i.e. functions $f\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$ chosen uniformly at random). So, the method of proof demonstrates a fairly general method for proving other concentration inequalities on random trees. This method is notable since random trees do not exhibit much independence, whereas uniformly random mappings do. That is, this technique allows the transfer of a concentration inequality from a setting of independent random variables, to a non-independent setting.

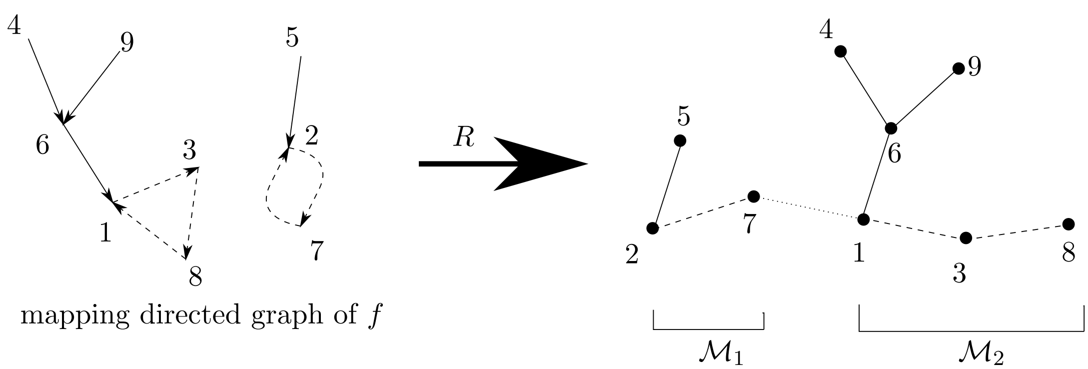

FIGURE 5. Example of the Rényi-Joyal bijection R, used to prove a concentration inequality on random trees. The left picture is the mapping directed graph of a function $f\colon \{1,\ldots,9\} \to \{1,\ldots,9\}$ where $f(1) = 3$, $f(3) = 8$, and so on. The two cyclic components of f are denoted as $\mathcal{M}_1 = \{2,7\}$ and $\mathcal{M}_2 = \{1,3,8\}$. Then $R(f)$ is the tree on the right.

8.3. **Erdős–Rényi random graphs.** The expected degree of a typical vertex in a random tree with n vertices is about 2, while the largest degree is about $\log n/(\log\log n)$. This typically small but occasionally large distribution of vertex degrees contributes to the difficulty of answering Question 12. For example, there are many independent sets of size comparable to the number n of vertices in the random tree. On the other hand, if the degrees of a different random graph are typically large and more evenly distributed, then Question 12 becomes easier to answer.

Let $V = \{1, \ldots, n\}$, $0 < p < 1$. Let E be a random subset of $\{\{i, j\} \in V \times V : i \neq j\}$ such that

$$\mathbb{P}(\{i, j\} \in E) = p, \qquad \forall\, 1 \leq i < j \leq n$$

and such that the events $\{\{i, j\} \in E\}_{1 \leq i < j \leq n}$ are independent. Then $G = (V, E)$ is an **Erdős–Rényi** random graph on n vertices with parameter $0 < p < 1$. This random graph is sometimes denoted as $G = G(n, p)$, and any vertex of $G(n, p)$ has expected degree $p(n - 1)$.

For Erdős–Rényi random graphs of large degree, Question 12 does have a positive answer.

Theorem 15 (**Erdős–Rényi Unimodality, Sparse Case, High Degree**). *Let $\varepsilon > 0$. Then for any $d \geq 10^{10/\varepsilon}$, $\exists\, c > 0$ such that, with probability at least $1 - e^{-cn}$, $G \in G(n, d/n)$ satisfies*

$$x_0(G) < x_1(G) < \cdots < x_{\lfloor \beta(1-\varepsilon)/2 \rfloor}(G), \quad \text{and} \quad x_{\lfloor \beta(1+\varepsilon)/2 \rfloor}(G) > \cdots > x_{\beta-1}(G) > x_\beta(G),$$

where β is the expected size of the largest independent set in $G(n, d/n)$. (It is known [Fri90] *that $\beta \approx (2/d)(\log d - \log\log d - \log 2 + 1)$.)*

8.4. **Potential Developments.** Theorems 14 and 13 were improved slightly in [BG21]. In both of our works, the following deterministic lower bound was used for the number of independent sets of fixed size in a tree. For any tree T on n vertices,

$$x_k(T) \geq \binom{n-k+1}{k}.$$

This inequality is an equality only when T is a single path. It seems reasonable that a better (i.e. larger) lower bound on $x_k(T)$ holds with high probability for a random tree T. If one could prove such a result, then Theorem 14 could be improved.

9. Strong Contractivity and Kahn-Kalai-Linial

9.1. **Discrete Analysis and Hypercontractivity.** Expander graphs, i.e. graphs with bounded degrees and large spectral gaps, have been studied extensively in both pure and applied mathematics [HLW06]. In the paper [MN14], the authors construct a family of graphs that have a spectral gap with respect to any uniformly convex Banach space. That is, these graphs are expander graphs in a much stronger sense than the usual definition of expander graphs. In order to improve their expander graph construction, Mendel and Naor made a conjecture concerning the decay of the heat semigroup in L_p spaces. The conjecture of [MN14] can be understood as an attempt to develop Littlewood-Paley theory for non-doubling metric spaces. For simplicity, we state this conjecture only in the case of real-valued functions.

Let n be a positive integer. Let $f : \{-1, 1\}^n \to \mathbb{R}$ be a function. Let μ be the uniform probability measure on the hypercube, so that $\mu(x) = 2^{-n}$ for each $x \in \{-1, 1\}^n$. Any $f : \{-1, 1\}^n \to \mathbb{R}$ can be written as $f = \sum_{S \subseteq \{1,\ldots,n\}} \widehat{f}(S) W_S$, where for all $x = (x_1, \ldots, x_n) \in \{-1, 1\}^n$, $W_S(x) := \prod_{i \in S} x_i$ and $\widehat{f}(S) := \int_{\{-1,1\}^n} f(x) W_S(x) d\mu(x)$. For any $t \geq 0$, define $e^{-tL} f := \sum_{S \subseteq \{1,\ldots,n\}} e^{-t|S|} \widehat{f}(S) W_S$, $Lf := \sum_{S \subseteq \{1,\ldots,n\}} |S|\, \widehat{f}(S) W_S$, and $\forall\ 1 \leq p < \infty$, define $\|f\|_p := (\int_{\{-1,1\}^n} |f(x)|^p\, d\mu(x))^{1/p}$. For all $i \in \{1, \ldots, n\}$, define the **influence** $I_i f$ **of the** i^{th} **variable on** f by

$$I_i f := \sum_{S \subseteq \{1,\ldots,n\} :\, i \in S} (\widehat{f}(S))^2.$$

Let $k \geq 1$, $k \in \mathbb{Z}$. The Conjecture [MN14, Remark 5.5] says: $\forall\ p > 1$, $\exists\ c(p) > 0$ such that

$$\widehat{f}(S) = 0\ \forall\, S \subseteq \{1, \ldots, n\} \text{ with } |S| < k \quad \Longrightarrow \quad \forall\, t > 0, \quad \|e^{-tL} f\|_p \leq e^{-tkc(p)} \|f\|_p. \tag{20}$$

Equivalently, Conjecture (20) is a **"higher order" Poincaré inequality**: $\forall\ p > 1$, $\exists\ c(p) > 0$ such that

$$\widehat{f}(S) = 0\ \ \forall\, S \subseteq \{1,\ldots,n\} \text{ with } |S| < k \quad \Longrightarrow \quad \int |f|^{p-1}\operatorname{sign}(f) Lf d\mu \geq kc(p)\int |f|^p\, d\mu.$$

A weaker form of (20) with the term $e^{-tkc(p)}$ replaced by $e^{-\min(t,t^2)kc(p)}$ can be proven [MN14, Lemma 5.4] using Hölder's inequality and the **hypercontractive inequality** [Sta59, Fed69, Bon70, Nel73, Gro75, Bec75]:

$$\forall\ 1 < p < q < \infty, \quad \forall\, t > \frac{1}{2}\log\left(\frac{q-1}{p-1}\right), \qquad \|e^{-tL}f\|_q \leq \|f\|_p. \tag{21}$$

Using the hypercontractive inequality (21), Kahn, Kalai and Linial proved the following famous inequality [KKL88], resolving a conjecture of Ben-Or and Linial.

Theorem 16 (**Kahn-Kalai-Linial**). [KKL88, Theorem 3.1] *There exists a universal constant $c > 0$ such that, $\forall\ f\colon \{-1,1\}^n \to \{-1,1\}$, we have* $\max_{i=1,\ldots,n} I_i f \geq c(\int[f - \int f d\mu]^2 d\mu)(\log n)/n$.

This Theorem says that a discrete function with values in $\{-1,1\}$ must have some asymmetry in its Fourier coefficients. To see this, note that it is easy to construct a function $f\colon \{-1,1\}^n \to \mathbb{R}$ such that $\max_{i=1,\ldots,n} I_i f \leq 10(\int[f - \int f d\mu]^2 d\mu)/n$, just by choosing f such that $\widehat{f}(S) = 2/(n(n-1))$ for all $|S| = 2$ and such that $\widehat{f}(S) = 0$ for all other $S \subseteq \{1,\ldots,n\}$. Note that the Fourier coefficients of f are then symmetric with respect to permutations on the set $\{1,\ldots,n\}$, but f does not take values $\{-1,1\}$, so Theorem 16 does not apply.

9.2. **Our Contribution.** In [HMO14], together with Mossel and Oleszkiewicz, we prove the case $k = 1$ of the Conjecture (20) of Mendel and Naor. In fact, we prove (20) for any probability space with a symmetric Markov semigroup P_t whose generator $L := -\frac{d}{dt}P_t|_{t=0^+}$ satisfies an L_2 Poincaré inequality. We then answer a question of Hatami and Kalai, showing that Theorem 16 cannot be strengthened unless a logarithmic number of Fourier coefficients of the function f vanish. That is,

Theorem 17 ([HMO14]). *There exists $c > 0$ such that, for all $n \in \mathbb{N}$, there exists $f\colon \{-1,1\}^n \to \{-1,1\}$ with $\widehat{f}(S) = 0$ for all $S \subseteq \{1,\ldots,n\}$ with $|S| \leq \log n$ such that* $\max_{i=1,\ldots,n} I_i f \leq c(\log n)/n$.

In the case that $\widehat{f}(S) = 0$ for all $S \subseteq \{1,\ldots,n\}$ with $|S| \leq C(n)\log n$, the equality $\sum_{i=1}^n I_i f = \sum_{S\subseteq\{1,\ldots,n\}} |S|\,(\widehat{f}(S))^2$ shows that $\max_{i=1,\ldots,n} I_i f \geq C(n)(\log n)/n$. So, there is a phase transition in the possible behavior of the maximum influence $\max_{i=1,\ldots,n} I_i f$, which occurs when $C(n) > 0$ is bounded or unbounded as $n \to \infty$.

Finally, we demonstrated a generalization of Talagrand's inequality for functions $f\colon \{-1,1\}^n \to \{-1,1\}$ with $\widehat{f}(S) = 0$ for all $S \subseteq \{1,\ldots,n\}$ with $|S| < k$. The usual Talagrand inequality then corresponds to the case $k = 1$.

Theorem 18 ([HMO14]). *Let $k \geq 1$. Let $f\colon \{-1,1\}^n \to \mathbb{R}$ with $\widehat{f}(S) = 0$ for all $S \subseteq \{1,\ldots,n\}$ with $|S| < k$. $\forall\ i = 1,\ldots,n$, let $\frac{\partial}{\partial x_i}f(x) := [f(x_1,\ldots,x_n) - f(x_1,\ldots,x_{i-1},-x_i,x_{i+1},\ldots,x_n)]/2$, where $x = (x_1,\ldots,x_n) \in \{-1,1\}^n$. Then*

$$\|f\|_2^2 \leq 6\sum_{i=1}^n \frac{\|\frac{\partial}{\partial x_i}f\|_2^2}{k + \log\left(\|\frac{\partial}{\partial x_i}f\|_2/\|\frac{\partial}{\partial x_i}f\|_1\right)}. \tag{22}$$

9.3. **Potential Developments.** Proving the case $k > 1$ of the Conjecture (20) of Mendel and Naor would give improved understanding of analysis in non-Euclidean spaces, going beyond (or supplementing) Littlewood-Paley theory in non-doubling metric measure spaces. Also, the solution of this problem would improve our understanding of expander graphs. Recent partial progress on Conjecture 20 was announced in [EI20], though Conjecture 20 remains largely open at present.

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