Research Statement, Steven Heilman

I am interested in applying analytic techniques to probabilistic problems. These problems are sometimes motivated by hardness results in theoretical computer science. A recurring theme of our research is that mathematical objects that were initially motivated by physics (such as semigroups of operators, hypercontractivity, Fourier analysis, soap bubbles and minimal surfaces, Bell's inequality from quantum mechanics, optimization of energy functionals, the calculus of variations, etc.) now have renewed motivation from important problems in computer science.

For example, one main focus of my research has been isoperimetric problems in Euclidean space equipped with the Gaussian measure [HJN13, Hei14, HMN16, Hei15, Hei20c, Hei17, Hei18, Hei19]. Recently, I proved a stability version of the Gaussian multi-bubble problem [Hei19]. This question asks for the partition of Euclidean space into $k \geq 3$ sets of fixed Gaussian volumes with smallest total Gaussian surface area. I showed that if a partition has nearly minimal Gaussian surface area, then this partition is close to the optimal partition. This result [Hei19] improves upon the solution of the Gaussian multi-bubble problem in [MN18b], where the optimal partition is identified. Also, [Hei19] implies previously unobtained applications to voting [IM12, Hei20a]. The results [MN18b, Hei19] were preceded by a sequence of work including [MN18a, Hei18] where $k = 3$ and $k = 4$ were treated, respectively. This problem was open since at least the 1990s [SM96, Problem 2] [Hut97]. The case $k = 2$ has been solved since the 1970s [SC74]. The solution of the "double-bubble problem" for the Lebesgue measure, i.e. finding two disjoint Euclidean sets with minimal total Euclidean surface area was resolved in a well-known 2002 result [HMRR02]. The case of three or more sets remains elusive, despite the recent progress on the Gaussian problem.

Recently [Hei17, Hei19], I also essentially resolved a question of Barthe from 2001 [Bar01]. This question asks for the symmetric Euclidean set of fixed Gaussian volume with smallest Gaussian surface area. The works [Hei17, Hei19] answers this question, with a small range of exceptions. In the work [Hei17] I used tools subsequently used in [Hei18].

I also solved [Hei20c] the endpoint case of a conjecture of Khot and Moshkovitz related to the Unique Games Conjecture, less a small error. The Unique Games Conjecture is a contemporary proxy for the P versus NP problem. So, my result [Hei20c] gives indirect evidence for an important conjecture in computational complexity theory.

A key aspect of the results [Hei17, Hei20c, Hei18, Hei19] is that they are dimension independent. That is, the proof techniques and results hold simultaneously for Euclidean space of any dimension.

My long term goal is to develop general methods for approaching a wide class of Gaussian isoperimetric problems, since such general methods did not exist previously. Most recently, I have been developing calculus of variations techniques [Hei20c, Hei18]. These techniques previously had little to no usage in this area.

Currently, there are many unsolved Gaussian isoperimetric problems, and their resolution will yield countless dividends. The Gaussian measure is most interesting since it is almost interchangeable with the uniform measure on the discrete hypercube, via Central Limit Theorems [Rot79, Cha06, MOO10, Mos10, IM12]. These Gaussian isoperimetric results therefore imply inequalities on the discrete hypercube. The discrete inequalities are then applied to machine learning and Grothendieck inequalities [KN09, KN13, HV16], to the Unique Games Conjecture [KKMO07, MOO10, KM16], to semidefinite programming algorithms such as MAX-CUT [KKMO07, IM12], to social choice theory [MOO10, IM12, Hei20a], to learning theory [FGRW12], to communication complexity [CR11], etc. So, solving these isoperimetric problems can tell us how quickly computers can run, and how to design elections so that erroneous tabulation of votes (or hacking) does not affect the outcome of the election.

Recently, I have worked on independent sets in random graphs and random trees [Hei20b, Hei20d]. An independent set of size k in a finite undirected graph is a set of k vertices of the graph, no two of which are connected by an edge. Let $x_k(G)$ be the number of independent sets of size k in a graph G and let $\alpha(G)=\max\{k\geq 0\colon x_k(G)\neq 0\}$. In 1987, Alavi, Malde, Schwenk and Erdös asked if the independent set sequence $x_0(G),x_1(G),\ldots,x_{\alpha(G)}(G)$ of a tree is unimodal (the sequence goes up and then down). This problem is still open. In 2006, Levit and Mandrescu showed that the last third of the independent set sequence of a tree is decreasing. We show [Hei20b, Hei20d] that the first 46.8% of the independent set sequence of a random tree is increasing with (exponentially) high probability as the number of vertices goes to infinity. So, the question of Alavi, Malde, Schwenk and Erdös is "four-fifths true", with high probability.

We also show unimodality of the independent set sequence of Erdös-Renyi random graphs, when the expected degree of a single vertex is large (with (exponentially) high probability as the number of vertices in the graph goes to infinity, except for a small region near the mode).

I have also studied L_p Poincaré inequalities for $1<p<\infty$ [HMO14] on general measure spaces that do not seem to be provable using standard techniques, such as Littlewood-Paley theory. In particular, we proved the degree one case of a conjecture of [MN14]. The conjectured Poincaré inequalities [MN14] can be used to construct graphs that are expanders in a very general sense [MN14]. The explicit construction of expander graphs are then used in computer science [HLW06].

1. The Standard Simplex Conjecture

1.1. **Euclidean Isoperimetry.** Classical isoperimetry can be traced to ancient times, though its full understanding still remains incomplete. Generally speaking, we look for an object with least surface area among all objects of fixed volume. And we expect that the smallest surface area object has a simple structure.

Isoperimetric problems and minimal surface theory have a long history in differential geometry [Ste38, Sch72, Min96, Wei27, Hur02, Lév51, Sim68, Law70, Bor75, Alm76, Gro83, Sim83, BdC84, EH89, Tal95, BL96, HMRR02, CM12, CIMW13], with some motivation from the physics of soap bubbles [Pla73, Tay76]. In particular, soap bubbles we encounter in the real world are minimal surfaces. Many deep results have come from these investigations, and we hope to continue this tradition.

To see that our knowledge is still limited, note that we still do not know the three disjoint sets of fixed Euclidean volume with minimum total surface area. This problem is known as a triple bubble, or multi bubble problem [HMRR02, Rei08, CCH$^+$08]. As Hutchings writes on his website[1], "The triple bubble problem in $\mathbb{R}^3$ currently seems hopeless without some brilliant new idea." Recently, I essentially solved the Gaussian version of this problem [Hei18]. The Gaussian case for an arbitrary number of sets was then resolved in [MN18b]. A generalization of this problem, known as the Standard Simplex Conjecture, is still open (see (7) below). I am currently working on this more general problem [Hei19]. Gaussian isoperimetric problems have gained recent interest due to their applications in theoretical computer science (see Section 1.5). For mathematical background motivation, we begin with the usual Euclidean isoperimetric inequality.

Let $n\geq 1$ be an integer, let $A\subseteq\mathbb{R}^n$ be a Borel set with smooth boundary ∂A. Let $\mathrm{vol}_n(A)$ denote the Euclidean volume of A, and let $\mathrm{vol}_{n-1}(\partial A)$ denote the Euclidean surface area of ∂A. Let $r>0$ and let $B(0,r):=\{(x_1,\ldots,x_n)\in\mathbb{R}^n\colon x_1^2+\cdots+x_n^2\leq r^2\}$ be a Euclidean ball such that $\mathrm{vol}_n(A)=\mathrm{vol}_n(B(0,r))$. The **Classical Isoperimetric Inequality** says that the Euclidean ball has the smallest boundary among all sets of fixed volume:

$$\mathrm{vol}_n(A)=\mathrm{vol}_n(B(0,r))\quad\Longrightarrow\quad\mathrm{vol}_{n-1}(\partial A)\geq\mathrm{vol}_{n-1}(\partial B(0,r)).\tag{1}$$

Let $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$, and recall that $\|x\|_2:=(x_1^2+\cdots+x_n^2)^{1/2}=\langle x,x\rangle^{1/2}$. Let dx be Lebesgue measure on $\mathbb{R}^n$, and let $\gamma_n(x):=e^{-\|x\|_2^2/2}(2\pi)^{-n/2}$. Let $f\colon\mathbb{R}^n\to\mathbb{R}$ be a bounded (measurable) function and let $t\geq 0$. Let $\Delta:=-\sum_{i=1}^n\partial^2/\partial x_i^2$. Let $e^{-t\Delta}f(x)$ denote the **classical**

[1] math.berkeley.edu/~hutching/pub/bubbles.html

heat semigroup applied to f. That is, for any $x \in \mathbb{R}^n$,

$$e^{-t\Delta} f(x) := \int_{\mathbb{R}^n} f(x + y\sqrt{2t}\,) d\gamma_n(y).$$

The classical isoperimetric inequality (1) is a consequence of the following inequality for the heat semigroup $e^{-t\Delta}$, which can be proven via symmetrization [Led96].

$$\text{(2)} \quad \text{vol}_n(A) = \text{vol}_n(B(0,r)) \quad \Longrightarrow \quad \forall\, t \geq 0, \quad \int_{\mathbb{R}^n} 1_A \cdot e^{-t\Delta} 1_A dx \leq \int_{\mathbb{R}^n} 1_{B(0,r)} \cdot e^{-t\Delta} 1_{B(0,r)}\, dx.$$

So, as heat flows out of a given set, the one that preserves the most of its heat is the ball. To get (1) from (2), let $t \to 0$ in (2). The quantity $\int_{\mathbb{R}^n} 1_A \cdot e^{-t\Delta} 1_A dx$ is sometimes called the heat content of A. We will focus on statements of the form (2) below.

1.2. **Gaussian Isoperimetry.** Let $A \subseteq \mathbb{R}^n$ be a Borel set. Denote $\gamma_n(A) := \int_A \gamma_n(x) dx$. Let $H \subseteq \mathbb{R}^n$ be a **half space**. That is, H is the region that lies on one side of a hyperplane. The **Gaussian Isoperimetric Inequality** [SC74] says that the half space H has the smallest Gaussian surface area among all sets of fixed Gaussian volume, i.e.

$$\text{(3)} \qquad \gamma_n(A) = \gamma_n(H) \quad \Longrightarrow \quad \gamma_{n-1}(\partial A) \geq \gamma_{n-1}(\partial H).$$

Here, we denote the **Gaussian surface area** of ∂A by

$$\gamma_{n-1}(\partial A) := \liminf_{\varepsilon \to 0} (2\varepsilon)^{-1} \gamma_n(x \in \mathbb{R}^n \colon \exists\, y \in \partial A, \|x - y\|_2 < \varepsilon).$$

The result (3) of [SC74] has been elucidated and strengthened over the years [Bor85, Led94, Led96, Bob97, BS01, Bor03, MN15a, MN15b, Eld15, MR15, BBJ17].

For applications to theoretical computer science, it is more useful to have a Gaussian version of (2). To this end, we first replace the heat semigroup by the Ornstein-Uhlenbeck semigroup. Define the operator L by $Lf(x) := \Delta f(x) + \langle x, \nabla f(x) \rangle$, for any $x \in \mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $t \geq 0$, define the **Ornstein-Uhlenbeck semigroup** applied to $f \colon \mathbb{R}^n \to \mathbb{R}$ by

$$e^{-tL} f(x) := \int_{\mathbb{R}^n} f(xe^{-t} + y\sqrt{1 - e^{-2t}}\,) d\gamma_n(y).$$

Then Borell's inequality [Bor85], which can be proven by symmetrization [BS01], says

$$\text{(4)} \qquad \gamma_n(A) = \gamma_n(H) \quad \Longrightarrow \quad \forall\, t \geq 0, \quad \int_{\mathbb{R}^n} 1_A \cdot e^{-tL} 1_A d\gamma_n \leq \int_{\mathbb{R}^n} 1_H \cdot e^{-tL} 1_H d\gamma_n.$$

One can deduce (3) from (4), by letting $t \to 0^+$ in (4). The quantity $\int_{\mathbb{R}^n} 1_A \cdot e^{-tL} 1_A d\gamma_n$ is referred to as the **noise stability** of the set A with parameter e^{-t}. Borell's original result (4) from [Bor85] has been highly influential, and we wish to duplicate his success by generalizing (4). Inequality (4) should be considered well-understood, due to recent proofs [MN15b, Eld15] of stability versions of (4). That is, the inequality (4) is close to equality if and only if A is close to a half space.

1.3. **Differential Geometry and the Colding-Minicozzi Theory for Gaussian Minimal Surfaces.** In a landmark investigation of mean curvature flow [CM12], Colding and Minicozzi studied a maximal version of the Gaussian surface area of an $(n-1)$-dimensional hypersurface $\Sigma \subseteq \mathbb{R}^n$. They called this translation and dilation-invariant quantity

$$\text{(5)} \qquad \sup_{a > 0, b \in \mathbb{R}^n} \int_\Sigma a^{-\frac{n-1}{2}} \gamma_n((x - b)a^{-1/2}) dx$$

the "entropy" of Σ. The Colding-Minicozzi entropy (5) is of interest since it monotonically decreases under the mean curvature flow. For this reason, [CM12] studied (local) minimizers of (5). In the context of mean curvature flow, the Colding-Minicozzi entropy (5) is an analogue of Perelman's reduced volume for Ricci flow.

As I demonstrated in [Hei17, Hei18], the methods of [CM12] also apply to the Gaussian surface area functional itself. Such a connection did not previously appear in the literature.

The Colding-Minicozzi theory studies eigenfunctions of the Ornstein-Uhlenbeck type operator

$$\Delta_\Sigma + \langle x, \nabla_\Sigma \rangle - \|F\|_2^2 - 1, \qquad \forall\, x \in \Sigma \subseteq \mathbb{R}^n, \tag{6}$$

associated to the surface $\Sigma := \partial A$. Here $\Delta_\Sigma, \nabla_\Sigma$ are the Laplacian and gradient, respectively, on the surface Σ. Also, $F = F_x$ is the **second fundamental form** of Σ at x, i.e. F is the matrix of first order partial derivatives of the unit normal vector at $x \in \Sigma$, and $\|F\|_2^2$ is the sum of the squares of the entries of F.

It was conjectured in [CIMW13] and ultimately proven using the Colding-Minicozzi theory in [Zhu20] that, among all compact $(n-1)$-dimensional hypersurfaces $\Sigma \subseteq \mathbb{R}^n$ with $\partial\Sigma = \emptyset$, the round sphere minimizes the quantity (5).

The idea of studying the eigenfunctions of an operator restricted to a minimal surface seems to be due to Simons [Sim68].

1.4. **Social Choice Theory.** By combining (4) with the invariance principle (12) below, the work [MOO10] solved the Majority is Stablest problem. This problem says that the most noise-stable way to determine the winner of an election between two candidates is to take the majority. This result assumes that no one person has too much influence over the election's outcome. The rigorous statement of this problem uses functions with domain $\{-1, 1\}^n$ (see Section 6.1). The corresponding statement for more than two voters, the Plurality is Stablest Problem, would follow from Conjecture (7) below. For this reason, Conjecture (7) below is of interest. The Plurality is Stablest Problem says that the most noise-stable way to determine the winner of an election between $k > 2$ candidates is to take the plurality. This result assumes that no one person has too much influence over the election's outcome [IM12, Hei20a].

Applications of mathematics to the analysis of elections arguably began with Marquis de Condorcet in the 1700s, with further developments by Game Theorists such as Shapley, Shubik and Banzhaf in the 1950s and 1960s [SS54, Ban65]. In the last three decades, discrete Fourier analysis has provided new insights into social choice theory for mathematics and computer science [KKL88, Kal02, MOO10, Mos12]. Yet, the Plurality is Stablest Problem for elections with three or more candidates is still unresolved [IM12], since the three set version of (4) is still unresolved (see (7) below when $k = 3$).

1.5. **Computational Complexity of Clustering Algorithms: MAX-k-CUT.** Borell's inequality (4) gives a sharp computational hardness result for the MAX-CUT problem [KKMO07]. The MAX-CUT problem asks for the partition of the vertices of an undirected graph into two disjoint sets that maximizes the number of edges going between the two sets. One can find a partition of the vertices achieving .87856 times the maximum number of cut edges in polynomial time [GW95]. And assuming the Unique Games Conjecture, the constant .87856 is the largest possible number for which the previous sentence holds. When we modify the MAX-CUT problem to allow a partition of the vertices of the graph into $k > 2$ disjoint sets, we get the MAX-k-CUT problem. And for this problem, there is only a conjecture for the best possible approximation that can be done in polynomial time. More specifically, there is a polynomial time algorithm that finds a partition of vertices of some constant c_k times the maximum number of cut edges. And if (7) below is true, then the existence of a polynomial time algorithm achieving a guaranteed fraction of cut edges larger than c_k would violate the Unique Games Conjecture [IM12]. (We already know that $c_2 = .87856$ [KKMO07].) So, Conjecture (7) below tells us the best possible way to cluster graphs into different pieces. The MAX-k-CUT problem can be considered a clustering problem, since it allows data to be clustered into disjoint pieces (e.g. we could consider a two data points to be connected by an edge if we judge the data points to be dissimilar). Algorithms for MAX-k-CUT have also been applied to the community detection problem for graphs [AS15, ABKK17, AL18, HWX16, MPW16].

1.6. **Gaussian Isoperimetry for multiple sets.** The endpoint case of the Standard Simplex Conjecture asks for the minimum total Gaussian surface area of a partition of $\mathbb{R}^n$ into $k > 2$ sets, each of Gaussian measure $1/k$. The full conjecture, stated in (7), does not seem to follow from a symmetrization argument [BS01, IM12], or from the methods of [MN15b, Eld15]. However, the calculus of variations methods we have developed [Hei17, Hei18] might apply to this problem

Let $A_1, \ldots, A_k \subseteq \mathbb{R}^n$ with $3 \leq k \leq n+1$, $n \geq 2$, $\cup_{i=1}^k A_i = \mathbb{R}^n$, $\gamma_n(A_i) = 1/k$. We now describe the conjectured maximizer of noise stability. Let $z_1, \ldots, z_k \in \mathbb{R}^n$ be the vertices of a regular k-simplex, which is centered at the origin of $\mathbb{R}^n$. For any $i = 1, \ldots, k$, let $B_i := \{x \in \mathbb{R}^n : \langle x, z_i\rangle = \max_{j=1,\ldots,k}\langle x, z_j\rangle\}$. Then $\{B_i\}_{i=1}^k$ is a partition of $\mathbb{R}^n$ into k regular simplicial cones. Generalizing (4), the **Standard Simplex Conjecture** says

$$\begin{aligned}&\gamma_n(A_i) = 1/k,\, \forall\, i = 1, \ldots, k \quad \wedge \quad \cup_{i=1}^k A_i = \mathbb{R}^n\\ (7)\qquad &\Longrightarrow \quad \forall\, t \geq 0, \quad \sum_{i=1}^k \int_{\mathbb{R}^n} 1_{A_i} \cdot e^{-tL} 1_{A_i} d\gamma_n \leq \sum_{i=1}^k \int_{\mathbb{R}^n} 1_{B_i} \cdot e^{-tL} 1_{B_i} d\gamma_n.\end{aligned}$$

Morally speaking, the results of [Eva93] imply that, if equality holds in (7), then for all $i = 1, \ldots, k$, the set ∂A_i should have constant mean curvature, except on a negligible subset. It is difficult to turn this intuition into a proof (unless we know ahead of time that the same sets optimize noise stability in (7) for all $t > 0$ [MS02]), but this intuition explains why the Standard Simplex Conjecture (7) is believed to be true.

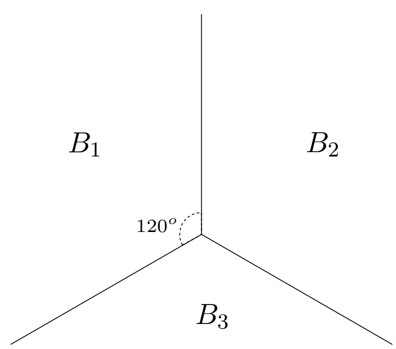

Figure 1. Optimal Sets for Conjecture (7) in the case $k = 3$, $n = 2$.

1.7. **Our Contribution.** Recently, I proved a stability version of the endpoint case $t \to 0^+$ of Conjecture (7) when $k \geq 3$ [Hei19]. The $k = 4$ case was entirely open since at least the 1990s [SM96, Problem 2] [Hut97]. The $k = 3$ case was partially resolved in [CCH$^+$08] and then recently resolved in [MN18a]. The case of an arbitrary number $k \geq 3$ of sets with $t \to 0^+$ in (7) was then solved in [MN18b]. I then improved this result in [Hei19] by showing that sets close to minimizing the total Gaussian surface area are close to the optimal sets. The analogue of the $k = 3$ result [Hei18, MN18a] for Lebesgue measure was resolved in a well-known 2002 result [HMRR02]. The analogue of the result of [MN18b] for Lebesgue measure with $k > 3$ sets was believed to be impossible, and perhaps it is now believed to be slightly less impossible.

I am currently working on adapting the argument of [Hei19] to large t and arbitrary $k \geq 3$ for Conjecture (7).

In an earlier result [Hei14], I showed the following. For any $n \geq 2$, there exists $t(n) > 0$ such that for any $t(n) < t < \infty$ and $k = 3$, the conjecture (7) holds. I used geometric and Fourier analytic arguments to show the first variation of (7) defines a contractive mapping, when restricted to partitions that almost achieve equality in (7).

In [HMN16], together with Mossel and Neeman, we showed that if the measure restriction of (7) is changed, then the most natural restatement of the conjecture (7) is false. Specifically, if $a_1, \ldots, a_k$ are real numbers with $0 < a_i < 1$ for all $i = 1, \ldots, k$ and $\sum_{i=1}^{k} a_i = 1$ with $(a_1, \ldots, a_k) \neq (1/k, \ldots, 1/k)$, and if $t > 0$, then the inequality (7) does not hold if we try to replace the sets $\{B_i\}_{i=1}^{k}$ with any set of simplicial cones that partition Euclidean space. This negative result implies that the endpoint case $t \to 0^+$ of (7) is quite different from the case $t > 0$, since the case $t \to 0^+$ holds for any measure restriction [MN18b] but the case $t > 0$ can only hold when all of the sets have equal Gaussian measures. Indeed, the proof of [MN18b] when $t \to 0^+$ proves their result by considering all possible measure restrictions simultaneously. My proof does not have this shortcoming [Hei18] so that it might have a better chance of applying to (7).

2. Barthe's Symmetric Gaussian Problem

Borell's inequality (4) [Bor85] implies (3): half spaces have the smallest Gaussian surface area among all Euclidean subsets of fixed Gaussian volume. Half spaces lie on one side of a hyerplane, so these sets are not symmetric with respect to reflection across the origin. If we try to instead minimize Gaussian surface area among *symmetric* sets of fixed Gaussian volume, it is a priori unclear what set is the best, since half spaces are now excluded. For this reason, Barthe [Bar01] posed Conjecture 1 below in 2001. I also planned to consider Conjecture 1 as a test case for developing methods for related questions with more applications such as (7) above. This plan succeeded, as the methods developed in [Hei15, Hei17] were then applied in my later work [Hei18].

2.1. Barthe's Symmetric Gaussian Problem.

Conjecture 1 ([Bar01])**.** *Let $A \subseteq \mathbb{R}^n$ have the smallest Gaussian surface area $\gamma_{n-1}(\partial A)$ among all sets of fixed Gaussian volume $\gamma_n(A)$, subject to the constraint $A = -A$. Then ∂A must be a* ***round cylinder****. That is, after applying a rotation to ∂A, $\exists\ r > 0$ and $\exists\ 0 \leq k \leq n-1$ such that $\partial A = rS^k \times \mathbb{R}^{n-k-1}$, where $S^k = \{(x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} \colon x_1^2 + \cdots + x_{k+1}^2 = 1\}$.*

If we remove the constraint $A = -A$ from Conjecture 1, then this problem is well understood by (3). However, all known proofs of (3) (with the exception of [MR15, BBJ17]) seem unable to handle the additional constraint that the set A is symmetric (i.e., that $A = -A$). In the work [Hei17], I demonstrate that the calculus of variations techniques of [CM12, MR15, BBJ17] succeed in this task, where other proof strategies seem insufficient.

2.2. Our Contribution: Gaussian Surface Area of Symmetric Sets.

We now state our main result in [Hei17]. Let $A \subseteq \mathbb{R}^n$ have minimal Gaussian surface area among all sets satisfying $A = -A$ with fixed Gaussian measure $\gamma_n(A)$. Let $F = F_x$ be the second fundamental form of ∂A at x. Let $\|F\|_2^2$ be the sum of the squares of the entries of F, and let $\|F\|_{2\to 2}$ denote the ℓ_2 operator norm of F.

It is shown in [Hei17, Hei19] that if either

$$\int_{\partial A} (\|F_x\|_2^2 - 1)\gamma_{n-1}(x)dx > 0 \qquad \text{or} \qquad \int_{\partial A} \Big(\|F_x\|_2^2 - 1 + 2\sup_{y\in\partial A} \|F_y\|_{2\to 2}^2\Big)\gamma_{n-1}(x)dx < 0,$$

then ∂A must be a round cylinder. That is, except for the case that the average value of $\|F\|_2^2$ is slightly less than 1, we resolve the convex case of Conjecture 1 from 2001.

In an earlier proof [Hei17], I required that A or A^c be convex, but this assumption was removed in [Hei19].

The main tool of [Hei17, Hei19] is the Colding-Minicozzi theory for Gaussian minimal surfaces described in Section 1. A key new ingredient is the use of a randomly chosen degree 2 polynomial in the second variation (i.e., second derivative) formula for the Gaussian surface area.

Using a slightly different "penalty function" approach, [BJ20] subsequently proved Conjecture 1 when the measure restriction is close to one (or zero).

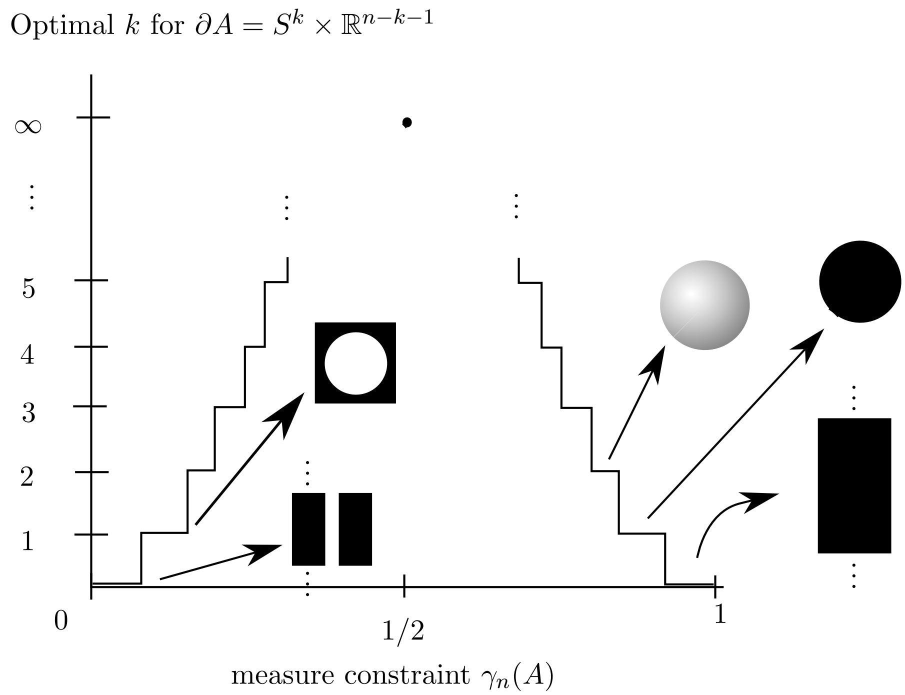

FIGURE 2. Frank Morgan has conjectured that the symmetric set of smallest Gaussian surface area has boundary $S^k \times \mathbb{R}^{n-k-1}$, where k is a function of the measure constraint as depicted here. So, Morgan's Conjecture refines Conjecture 1 above.

2.3. **Potential Developments.** In the near future, I hope to extend the main result of [Hei17, Hei19] to noise stability. That is, we want to prove the same result as [Hei17], but with noise stability (as in (4)) replacing the Gaussian surface area. This problem, Conjecture 2 below, was first mentioned in [CR11, O'D12] in relation to the Gap-Hamming-Distance problem from communication complexity. The methods of [Hei17] are fairly different from the earlier work [Hei15], where we computed the second variation of noise stability of the ball and its complement.

Also, it seems possible that the approach developed in [Hei17] could be further applied to provide a variational proof of (4). We have many proofs of (4) [Bor85, BS01, MN15b, Eld15], but none of these proofs are genuinely "local" in nature. Instead of adjusting a set $A \subseteq \mathbb{R}^n$ in a "local" way, proofs of (4) somehow globally replace the set $A \subseteq \mathbb{R}^n$ with another set of smaller Gaussian surface area. Consequently, these proofs cannot handle the symmetry constraint $A = -A$ in Problem 2.

Conjecture 2 (**Symmetric Gaussian Problem**, A Generalization of Conjecture 1)**.** *Let $t > 0$. Let $A \subseteq \mathbb{R}^n$ have the largest noise stability $\int_{\mathbb{R}^n} 1_A \cdot e^{-tL} 1_A d\gamma_n$ among all sets of fixed Gaussian volume $\gamma_n(A)$, subject to the constraint $A = -A$. Then ∂A must be a round cylinder.*

Below, we propose methods for extending the results of [Hei17] to other related problems discussed in Sections 1 and 4.

2.4. **Our Contribution: Local Version of Conjecture 2 for large t.** In an earlier work [Hei15], I show that Conjecture 2 is true for $n = 1$ and t sufficiently large, by simplifying the argument from my earlier work on the Standard Simplex Conjecture [Hei14] mentioned in Section 1. When $n \geq 2$, I compute the second variation of the ball and its complement, adapting a second variation argument from [CS07]. It turns out that the computation of the second variation of noise stability of the ball essentially reduces to proving an L_2 Poincaré inequality on the sphere.

3. The Unique Games Conjecture and Gaussian Isoperimetry

3.1. **The Unique Games Conjecture.** The Unique Games Conjecture [Kho02] is a standard assumption in theoretical computer science. This conjecture can be considered a contemporary proxy for the assumption that P≠NP. That is, proving or disproving the Unique Games Conjecture is expected to have similar significance and consequences to proving or disproving P≠NP. Moreover, both problems are closely related. As we will describe below, the Unique Games Conjecture can be succinctly stated as: approximate linear algebra is hard.

A recent breakthrough of [KMS18] gives significant positive evidence for The Unique Games Conjecture 4. In a certain sense, the result [KMS18] proves Conjecture 4 "half way." The remaining open cases of the Unique Games Conjecture 4 are closely related to Conjecture 6 discussed in the following section [Hei20c].

Definition 3 (**Γ-MAX-2LIN(p)**)**.** Let $p \geq 2$ be a prime number. We define the Γ-MAX-2LIN(p) problem. In this problem, we are given $n \in \mathbb{N}$ and $2n$ variables $x_i \in \mathbb{Z}/p\mathbb{Z}$, $i \in \{1, \ldots, 2n\}$. We are also given a matrix $\{a_{ij}\}_{i,j=1}^{2n}$ with $a_{ij} \geq 0$ for all $i, j \in \{1, \ldots, 2n\}$ and a set $E \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ with n elements. An element $(i, j) \in E$ corresponds to one of n linear equations of the form $x_i - x_j = c_{ij} (\mathrm{mod}\, p)$, where $c_{ij} \in \mathbb{Z}/p\mathbb{Z}$. The goal of the Γ-MAX-2LIN(p) problem is to find the following quantity:

$$\max_{(x_1,\ldots,x_{2n})\in(\mathbb{Z}/p\mathbb{Z})^{2n}} \sum_{\substack{(i,j)\in E\colon \\ x_i - x_j = c_{ij} (\mathrm{mod}\, p)}} a_{ij}. \tag{8}$$

That is, we need to maximize the (weighted) number of equations $x_i - x_j = c_{ij} (\mathrm{mod}\, p)$ that are satisfied.

Conjecture 4 (**Unique Games Conjecture**, [Kho02, KKMO07])**.** *For every $\varepsilon \in (0, 1)$, there exists a prime number $p(\varepsilon)$ such that no polynomial time algorithm (with respect to the parameter n) can distinguish between the following two cases, for instances of Γ-MAX-2LIN($p(\varepsilon)$) with $a_{ij} = 1$ for all $i, j \in \{1, \ldots, 2n\}$:*

- (i) (8) *is larger than* $(1 - \varepsilon)n$, *or*
- (ii) (8) *is smaller than* εn.

If (8) were equal to n, then we could find $(x_1, \ldots, x_{2n})$ achieving the maximum in (8) by Gaussian elimination. One can therefore interpret the Unique Games Conjecture as an assertion that approximate linear algebra is hard. The truth or falsity of this conjecture remains a major open problem.

3.2. **Khot and Moshkovitz's new approach.** Recently, [KM16] showed that a special case of the Unique Games Conjecture follows from the following conjecture. Recall that $v \in \mathbb{R}^n$ is a standard basis vector if one of its coordinates is 1 and all of its other coordinates are zero. Let $A \subseteq \mathbb{R}^n$ with $-A = A^c$ and such that $A + v = A^c$ for every standard basis vector $v \in \mathbb{R}^n$. That is, let A be a "periodic set." Then the noise stability of A is at most the noise stability of the set

$$H := \Big\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \colon\ \sin(\pi \sum_{i=1}^{n} x_i) \geq 0\Big\}.$$

Note that the boundary ∂H of H is the set of parallel hyperplanes of the form

$$\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \colon \exists\, k \in \mathbb{Z} \text{ such that } x_1 + \cdots + x_n = k\}.$$

For this reason, H is called a "periodic half space."

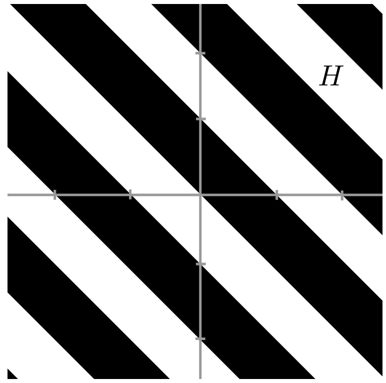

FIGURE 3. A periodic half space H.

Conjecture 5 ([KM16])**.** *Let $A \subseteq \mathbb{R}^n$ with $-A = A^c$ and such that $A + v = A^c$ for every standard basis vector $v \in \mathbb{R}^n$. Then for any $0 < t < 1$,*

$$\int_{\mathbb{R}^n} 1_A \cdot e^{-tL} 1_A d\gamma_n \leq \int_{\mathbb{R}^n} 1_H \cdot e^{-tL} 1_H d\gamma_n.$$

The case $t \to 0^+$ of Conjecture 5 is most relevant for the implications for the Unique Games Conjecture in [KM16]. So, as a first step, it is sensible to try to prove Conjecture 5 for the Gaussian surface area functional. That is, as a first step, we instead prove the following special case of Conjecture 5.

Conjecture 6 (Endpoint Case of Conjecture 5)**.** *Let $A \subseteq \mathbb{R}^n$ with $-A = A^c$ and such that $A + v = A^c$ for every standard basis vector $v \in \mathbb{R}^n$. Then*

$$\gamma_{n-1}(\partial A) \geq \gamma_{n-1}(\partial H).$$

3.3. **Our Contribution.** In [Hei20c], we prove Conjecture 6, less a small error.

Theorem 7 ([Hei20c])**.** *Let $A \subseteq \mathbb{R}^n$ with $-A = A^c$ and such that $A + v = A^c$ for every standard basis vector $v \in \mathbb{R}^n$. Then*

$$\gamma_{n-1}(\partial A) \geq (1 - 6 \cdot 10^{-9})\gamma_{n-1}(\partial H).$$

Actually, we also incorporate a "robustness" term that says if ∂A is far from ∂H, then $\gamma_{n-1}(\partial A)$ is much larger than $\gamma_{n-1}(\partial H)$. That is, we show in [Hei20c] that

$$\gamma_{n-1}(\partial A) \geq (1 - 6 \cdot 10^{-9})\gamma_{n-1}(\partial H) + \int_{\partial A} \Big(1 - \frac{\|N(x)\|_1}{\sqrt{n}}\Big)\gamma_n(x)dx.$$

Here $N(x)$ is a unit normal vector at $x \in \partial A$.

The proof is fairly elementary. Using the Poisson Summation Formula, Conjecture 6 can be rewritten as an isoperimetric problem on the torus equipped with the heat kernel measure. This heat kernel measure is very close to being constant. So, a Euclidean projection argument can then prove Theorem 7.

Using standard arguments, these results imply the following weak version of Conjecture 5.

Theorem 8 (Weak Version of Conjecture 5, [Hei20c])**.** *Let $A \subseteq \mathbb{R}^n$ with $-A = A^c$ and such that $A + v = A^c$ for every standard basis vector $v \in \mathbb{R}^n$. Then for any $0 < t < 1/2$,*

$$\int_{\mathbb{R}^n} 1_A \cdot e^{-tL} 1_A d\gamma_n \leq \int_{\mathbb{R}^n} 1_H \cdot e^{-tL} 1_H d\gamma_n + 3 \cdot 10^{-9} - \frac{\sqrt{1 - e^{-2t}}}{\sqrt{2\pi}} \int_{\partial\Omega} \Big(1 - \frac{\|N(x)\|_1}{\sqrt{n}}\Big)\gamma_n(x)dx + o(t).$$

Here the implied constant can depend on A. Note that if t is near zero then this bound is typically vacuous.

In summary, our results give indirect evidence for the truth of the Unique Games Conjecture, Conjecture 4.

3.4. **Potential Developments.** It is tempting to try to apply the variational methods of [BBJ17, Hei17, Hei18] to Conjecture 5. However, the error terms seem difficult to control, since many oscillatory terms appear. For example, if we try to use the Colding-Minicozzi theory from Section 1.3, we then must try to find the eigenfunctions of the operator

$$\Delta_{\Sigma} + \left\langle \frac{-\nabla \sum_{z\in\mathbb{Z}^n} \gamma_n(x+z)}{\sum_{z\in\mathbb{Z}^n} \gamma_n(x+z)}, \nabla_{\Sigma} \right\rangle - \|F\|_2^2 - 1, \qquad \forall\, x \in \Sigma = \partial A.$$

However, it seems difficult to identify or even approximate these eigenfunctions.

Even if the Unique Games Conjecture is eventually proven false, Conjectures 5 and 6 will still be of interest. Some similar isoperimetric problems on the torus are motivated by the physics of ferromagnetic materials, and these problems are predicted to have optimal sets consisting of parallel stripes, as in Conjectures 5 and 6. This "crystallization" prediction has been studied in e.g. [The06, BPT13, GM12, GS16, DR18], though these studies have typically focused only on $n = 2$ or $n = 3$, whereas Conjectures 5 and 6 are stated for any dimension $n \geq 1$. The isoperimetric problem on the flat torus equipped with the Lebesgue measure also appears to be unsolved [Ros01], so no methods from that problem seem relevant to Conjecture 6. Also, the problems studied in [The06, BPT13, GM12, GS16, DR18] still appear to be unsolved, so progress on Conjectures 5 and 6 could lead to progress on these other problems.

4. The Propeller Conjecture

4.1. **Invariance Principle.** The invariance principles of [Rot79, Cha06, MOO10] are nonlinear versions of the Central Limit Theorem, with error bounds. That is, the invariance principle implies the Berry-Esséen Central Limit Theorem, which we now recall.

Let n be a positive integer. Let $x_1, \ldots, x_n$ be commutative indeterminate variables, and let

$$Q(x_1, \ldots, x_n) := \frac{x_1 + \cdots + x_n}{\sqrt{n}}.$$

Let $b_1, \ldots, b_n$ be independent identically distributed (i.i.d.) uniform random variables in $\{-1, 1\}$, and let $g_1, \ldots, g_n$ be i.i.d. standard Gaussian random variables. Letting $\mathbb{E}$ denote expected value, we then define the 2-norm of Q to be

$$\|Q\|_2 := (\mathbb{E}\,|Q(b_1, \ldots, b_n)|^2)^{1/2}. \tag{9}$$

The **Berry-Esséen Central Limit Theorem** then says

$$\sup_{t\in\mathbb{R}} |\mathbb{P}(Q(b_1, \ldots, b_n) \leq t) - \mathbb{P}(Q(g_1, \ldots, g_n) \leq t)| \leq 3 \max_{i=1,\ldots,n} \left\| \frac{\partial}{\partial x_i} Q \right\|_2. \tag{10}$$

If the rightmost expression looks unfamiliar, note that $Q(g_1, \ldots, g_n)$ has a standard Gaussian distribution, and $\|\frac{\partial}{\partial x_i} Q\|_2 = 1/\sqrt{n}$ for all $i \in \{1, \ldots, n\}$. The proof of (10) can also be extended to moments of Q:

$$\left| \mathbb{E}\,|Q(b_1, \ldots, b_n)|^4 - \mathbb{E}\,|Q(g_1, \ldots, g_n)|^4 \right| \leq 240 \max_{i=1,\ldots,n} \left\| \frac{\partial}{\partial x_i} Q \right\|_2. \tag{11}$$

A similar statement can be made for higher moments of Q. The commutative invariance principle [Rot79, Cha06, MOO10] implies, among other things, that (11) holds for multilinear polynomials.

Let $d \in \mathbb{N}$. Let $Q(x_1, \ldots, x_n)$ be a multilinear polynomial of degree d, so that

$$Q(x_1, \ldots, x_n) = \sum_{S \subseteq \{1,\ldots,n\}\colon |S| \leq d} c_S \prod_{i \in S} x_i, \quad c_S \in \mathbb{R}, \forall S \subseteq \{1, \ldots, n\}.$$

Assume that $\|Q\|_2 \leq 1$. Then the **commutative invariance principle** [MOO10] says that

$$\left|\mathbb{E}\,|Q(b_1, \ldots, b_n)|^4 - \mathbb{E}\,|Q(g_1, \ldots, g_n)|^4\right| \leq 24 \cdot 10^d \max_{i=1,\ldots,n} \left\|\frac{\partial}{\partial x_i} Q\right\|_2. \tag{12}$$

The commutative invariance principle (12) in [MOO10] is proven by a combination of the Lindeberg replacement argument and the hypercontractive inequality [Sta59, Fed69, Bon70, Nel73, Gro75, Bec75] (see (17) below). That is, one replaces one argument of Q at a time, adding up the resulting errors and controlling them via the hypercontractive inequality. The invariance principle (12) has seen many applications [O'D, O'D14] in recent years. Here is a small sample of such applications and references: isoperimetric problems in Gaussian space and in the hypercube [MOO10, IM12], social choice theory, Unique Games hardness results [KKMO07, IM12], analysis of algorithms [BR15], random matrix theory [MP14], free probability [NPR10], optimization of noise sensitivity [Kan14], etc.

4.2. **Grothendieck's Inequality.** The first application of (12) we will describe is computational hardness for the commutative Grothendieck inequality [RS09]. For any $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{C}^n$, define $\langle x, y \rangle := \sum_{i=1}^n x_i \overline{y_i}$ and define $\|x\|_2 := \sqrt{\langle x, x \rangle}$. Let $\{a_{ij}\}_{i,j=1}^n$ be a real matrix. Proven first in 1953, **Grothendieck's Inequality** [Gro53, LP68, AN06, BMMN13] says there exists a constant $K > 0$ which does not depend on n or on $\{a_{ij}\}_{i,j=1}^n$ such that

$$\sup_{\substack{w_1,\ldots,w_n,r_1,\ldots,r_n \in \mathbb{R}^{2n-1} \\ \|w_i\|_2 = \|r_i\|_2 = 1, \forall i=1,\ldots,n}} \sum_{i,j=1}^n a_{ij} \langle w_i, r_j \rangle \leq K \cdot \sup_{u_1,\ldots,u_n,\nu_1,\ldots,\nu_n \in \{-1,1\}} \sum_{i,j=1}^n a_{ij} u_i \nu_j. \tag{13}$$

That is, for a general optimization problem (corresponding to the left side of (13)), it is possible to "round" the unit vectors w_i, r_i, $i = 1, \ldots, n$ to a one-dimensional set of unit vectors u_i, ν_i, $i = 1, \ldots, n$. And the weighted sum of inner products of the vectors does not decrease very much after we perform this rounding procedure. It is known that $K < \pi/(2\log(1+\sqrt{2}))$ [BMMN13], and that a rounding procedure can establish the best constant in Grothendieck's inequality [NR14]. However, it remains a major open problem to find this optimal rounding procedure and to find the smallest possible constant K in Grothendieck's inequality.

Finding the smallest possible constant K in (13) has several interpretations beyond mathematics. From the physics perspective, the best constant K in (13) is also the smallest constant in certain Bell inequalities in quantum mechanics [Pis12]. More specifically, Bell's inequality says that the smallest K possible in (13) satisfies $K > 1$. From the computer science perspective, assuming the Unique Games Conjecture (see Section 3.1), it is impossible to approximate the right side of (13), in time polynomial in n, within a multiplicative factor smaller than K, where K is the smallest possible constant in the inequality (13) [RS09]. Mathematically, (13) can be rewritten as a ratio between two tensor product norms. One could consider this breadth of interpretation as evidence for the difficulty and importance of finding the smallest possible value of K in (13).

In an effort to better understand Grothendieck's inequality (13), and to approximate the optima of kernel clustering problems from machine learning, Khot and Naor [KN09, KN13] investigated Grothendieck's inequality (13) for $\{a_{ij}\}_{i,j=1}^n$ that are positive semidefinite.

4.3. **Generalized Positive Semidefinite Grothendieck Inequalities.** Suppose $\{a_{ij}\}_{i,j=1}^n$ is a positive semidefinite matrix, i.e. a real symmetric matrix with all eigenvalues nonnegative. Let $\nu_1, \ldots, \nu_k \in \mathbb{R}^k$ with $k \geq 2$, and let $B = \{b_{ij}\}_{i,j=1}^k$ be the symmetric positive semidefinite matrix with $b_{ij} := \langle \nu_i, \nu_j \rangle$. Then the **Generalized Positive Semidefinite Grothendieck Inequality**

[KN13, Theorem 3.1, Theorem 3.3] says that there exists $C(B) > 0$ which does not depend on n or on $\{a_{ij}\}_{i,j=1}^{n}$ such that

$$\max_{\substack{w_1,\ldots,w_n \in \ell_2 \\ \|w_i\|_2 = 1,\, \forall\, i=1,\ldots,n}} \sum_{i,j=1}^{n} a_{ij} \langle w_i, w_j \rangle \leq \frac{1}{C(B)} \cdot \max_{\sigma \colon \{1,2,\ldots,n\} \to \{1,2,\ldots,k\}} \sum_{i,j=1}^{n} a_{ij} \langle \nu_{\sigma(i)}, \nu_{\sigma(j)} \rangle. \tag{14}$$

Instead of rounding the vectors w_i, $i = 1, \ldots, n$ to 1 or -1 as in Grothendieck's inequality (13), Khot and Naor prove (14) by rounding the vectors w_i, $i = 1, \ldots, n$ to k vectors ν_i, $i = 1, \ldots, k$. Also, the best constant $1/C(B)$ is found by finding the best rounding procedure.

4.4. **The Sharp Constant of Grothendieck Inequalities.** Let I_k denote the $k \times k$ identity matrix. In [KN09, KN13], it is shown that $C(B)$ can be found by solving a finite dimensional optimization problem, whose parameters depend on B. However, this optimization problem is non-convex in general, so standard methods cannot compute $C(B)$. The **Propeller Conjecture** guesses the value of $C(B)$ for $B = I_k$, $k \geq 4$:

$$C(I_k) := \sup_{\substack{A_1,\ldots,A_k \colon \cup_{i=1}^{k} A_i = \mathbb{R}^{k-1}, \\ \gamma_{k-1}(A_i \cap A_j) = 0,\, \forall\, i,j \in \{1,\ldots,k\},\, i \neq j}} \sum_{i=1}^{k} \left\| \int_{A_i} x d\gamma_{k-1}(x) \right\|_2^2 = \frac{9}{8\pi} = C(I_3). \tag{15}$$

The constant $C(I_3) = 9/(8\pi)$ is computed in [KN09] using Lagrange multipliers, but $C(I_4)$ does appear to be computable using this technique.

4.5. **Our Contribution.** Together with Naor and Jagannath, using some theoretical results and a brute force search, we give a computer-assisted proof of the Conjecture (15) in the case $k = 4$ [HJN13]. The analytic results use a connection between the maximization problem (15) and discrete harmonic maps into the sphere. Intuition derived from this connection allows several ad hoc arguments to rule out candidates for partitions that maximize (15).

4.6. **Potential Developments.** Solving (15) for all $k \geq 4$ would yield better understanding of semidefinite programming algorithms and potential insight into computing the best constant in Grothendieck's inequality (13).

5. Independent Sets in Random Graphs and Random Trees

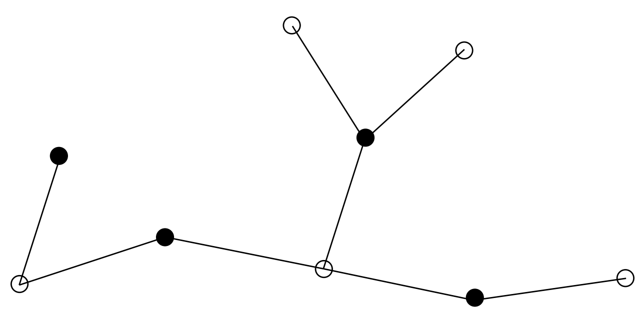

Figure 4. The open circles form an independent set of size 5 in the tree. The filled in circles form an independent set of size 4 in the tree.

Let G be a finite, undirected graph with no self-loops and no multiple edges, on $n \geq 1$ labelled vertices $V := \{1, \ldots, n\}$, with edges $E \subseteq \{\{i, j\} \colon i, j \in V,\, i \neq j\}$. Let $0 \leq k \leq n$. An **independent set** of size k in G is a subset of vertices no two of which are connected by an edge. Let $x_k = x_k(G)$ denote the number of independent sets in size k in G. (Note that $x_0(G) = 1$ since we consider the empty set to be a subset of $V = \{1, \ldots, n\}$.) We refer to the sequence $x_0(G), \ldots, x_n(G)$ as the **independent set sequence** of G.

Question 9. *What does the sequences $x_0, x_1, \ldots, x_n$ "look like" for random graphs?*

Some motivations for this question include:

- Statistical physics, where an independent set represents molecules in a magnet that do not want to be close to each other;
- Computer Science and Probability, where many combinatorial optimization problems (k-SAT, MAX-Independent Set, etc.) exhibit similar interesting behavior for random instances. See for example the phase transition known as "shattering" discussed in e.g. [CE15] and [DSS16].
- Combinatorics, where one would like to exactly or approximately find the numbers $x_0, \ldots, x_n$ for both deterministic and random graphs.

We note that a classic NP-complete problem is: For any $k \geq 1$ and any graph G, decide whether or not $x_k(G) > 0$. So, it could be hard for a computer to decide whether or not a large graph has a large independent set. Counting the *number* of independent sets of a given size is then computationally more difficult. In fact, a remarkable result of [JSV04] implies a computational equivalence between approximately counting combinatorial quantities (such as independent sets), and constructing a stochastic process whose distribution converges to the uniform distribution on those combinatorial objects.

Intuitively, $x_0, \ldots, x_n$ should resemble the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$. Note that if $E = \emptyset$, $x_k = \binom{n}{k}$ for all $0 \leq k \leq n$. The sequence of binomial coefficients is unimodal, so one might expect the independent set sequence of a random graph to have this same behavior.

5.1. **Independent Sets of Trees.** A **tree** on n vertices is a connected graph with no cycles. The following precise form of the rather vague Question 9 for trees was stated by Alavi, Malde, Schwenk and Erdös in 1987.

Question 10 ([AMSE87])**.** *Does every tree have a unimodal independent set sequence?*

Despite much effort, including [LM02, LM03, Zhu07, WZ11, Gal11, Gal12, BBO14, Zhu16, GH18], Question 10 remains open. The cited works mostly focus on answering Question 10 for particular families of trees. One general partial result towards Question 10 is the following.

Theorem 11 ([LM07])**.** *Let T be a tree whose largest independent set is of size j. Then the "last third" of the independence set sequence is unimodal:*

$$x_{\lceil (2j-1)/3 \rceil}(T) \geq x_{1+\lceil (2j-1)/3 \rceil}(T) \geq \cdots \geq x_{j-1}(T) \geq x_j(T).$$

That is, Question 10 is "one-third true."

Motivated by a question of Galvin, instead of trying to answer Question 10 for deterministic families of trees, we attempt to answer question two for random trees, with high probability.

A **random tree** T on $n \geq 2$ vertices is a random graph that is equal to any of the n^{n-2} possible labelled trees on n vertices, each with probability $1/n^{n-2}$.

5.2. **Our Contribution.**

Theorem 12 (**Partial Unimodality for Random Trees**, [Hei20b])**.** *There exists $c > 0$ such that, with probability at least $1 - e^{-cn}$, a random tree T on n vertices satisfies*

$$x_0(T) < x_1(T) < \cdots < x_{\lfloor (.26543)n \rfloor}(T).$$

For comparison, the largest independent set size in a random tree is about $.567143n$, with fluctuations of order $\sqrt{n}$, by the Azuma-Hoeffding inequality [Fri90]. So, the first 46.8% of the nontrivial independent set sequence is unimodal. Combined with Theorem 11 of Levit and Mandrescu, Question 10 is "four-fifths true", with high probability.

The proof of Theorem 12 required proving a concentration inequality for the number of neighbors in a random tree conditioned on having an independent set of fixed size [Hei20d]. The proof of this concentration inequality used a bijection between random trees and random mappings (i.e. functions $f\colon \{1,\ldots,n\}\to\{1,\ldots,n\}$ chosen uniformly at random). So, the method of proof demonstrates a fairly general method for proving other concentration inequalities on random trees. This method is notable since random trees do not exhibit much independence, whereas uniformly random mappings do. That is, this technique allows the transfer of a concentration inequality from a setting of independent random variables, to a non-independent setting.

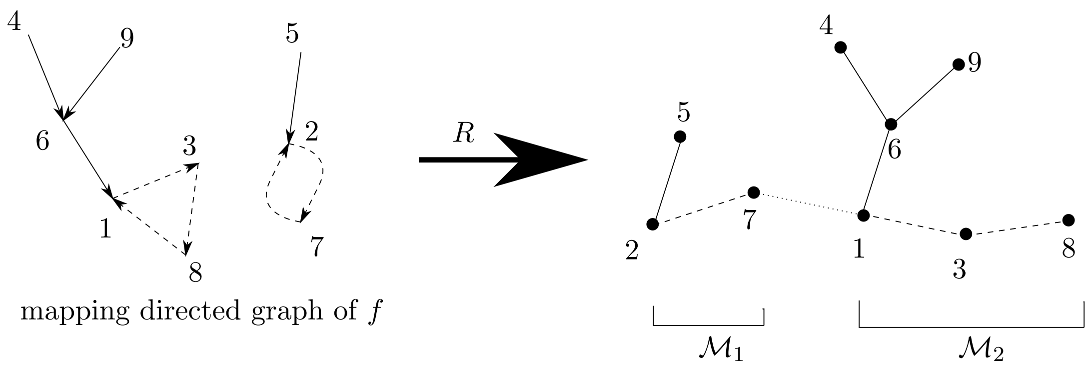

FIGURE 5. Example of the Rényi-Joyal bijection R, used to prove a concentration inequality on random trees. The left picture is the mapping directed graph of a function $f\colon\{1,\ldots,9\}\to\{1,\ldots,9\}$ where $f(1)=3$, $f(3)=8$, and so on. The two cyclic components of f are denoted as $\mathcal{M}_1=\{2,7\}$ and $\mathcal{M}_2=\{1,3,8\}$. Then $R(f)$ is the tree on the right.

5.3. **Erdös-Renyi random graphs.** The expected degree of a typical vertex in a random tree with n vertices is about 2, while the largest degree is about $\log n/(\log\log n)$. This typically small but occasionally large distribution of vertex degrees contributes to the difficulty of answering Question 10. For example, there are many independent sets of size comparable to the number n of vertices in the random tree. On the other hand, if the degrees of a different random graph are typically large and more evenly distributed, then Question 10 becomes easier to answer.

Let $V=\{1,\ldots,n\}$, $0<p<1$. Let E be a random subset of $\{\{i,j\}\in V\times V\colon i\neq j\}$ such that

$$\mathbb{P}(\{i,j\}\in E)=p,\qquad \forall\, 1\leq i<j\leq n$$

and such that the events $\{\{i,j\}\in E\}_{1\leq i<j\leq n}$ are independent. Then $G=(V,E)$ is an **Erdös-Renyi** random graph on n vertices with parameter $0<p<1$. This random graph is sometimes denoted as $G=G(n,p)$, and any vertex of $G(n,p)$ has expected degree $p(n-1)$.

For Erdös-Renyi random graphs of large degree, Question 10 does have a positive answer.

Theorem 13 (**Erdös-Renyi Unimodality, Sparse Case, High Degree**). *Let $\varepsilon>0$. Then for any $d\geq 10^{10/\varepsilon}$, $\exists\, c>0$ such that, with probability at least $1-e^{-cn}$, $G\in G(n,d/n)$ satisfies*

$$x_0(G)<x_1(G)<\cdots<x_{\lfloor\beta(1-\varepsilon)/2\rfloor}(G),\quad\text{and}\quad x_{\lfloor\beta(1+\varepsilon)/2\rfloor}(G)>\cdots>x_{\beta-1}(G)>x_\beta(G),$$

where β is the expected size of the largest independent set in $G(n,d/n)$. (It is known [Fri90] *that $\beta\approx(2/d)(\log d-\log\log d-\log 2+1)$.)*

5.4. **Potential Developments.** Theorems 12 and 11 were improved slightly in [BG20]. In both of our works, the following deterministic lower bound was used for the number of independent sets of fixed size in a tree. For any tree T on n vertices,

$$x_k(T)\geq\binom{n-k+1}{k}.$$

This inequality is an equality only when T is a single path. It seems reasonable that a better (i.e. larger) lower bound on $x_k(T)$ holds with high probability for a random tree T. If one could prove such a result, then Theorem 12 could be improved.

6. Strong Contractivity and Kahn-Kalai-Linial

6.1. **Discrete Analysis and Hypercontractivity.** Expander graphs, i.e. graphs with bounded degrees and large spectral gaps, have been studied extensively in both pure and applied mathematics [HLW06]. In the paper [MN14], the authors construct a family of graphs that have a spectral gap with respect to any uniformly convex Banach space. That is, these graphs are expander graphs in a much stronger sense than the usual definition of expander graphs. In order to improve their expander graph construction, Mendel and Naor made a conjecture concerning the decay of the heat semigroup in L_p spaces. The conjecture of [MN14] can be understood as an attempt to develop Littlewood-Paley theory for non-doubling metric spaces. For simplicity, we state this conjecture only in the case of real-valued functions.

Let n be a positive integer. Let $f\colon \{-1,1\}^n \to \mathbb{R}$ be a function. Let μ be the uniform probability measure on the hypercube, so that $\mu(x) = 2^{-n}$ for each $x \in \{-1,1\}^n$. Any $f\colon \{-1,1\}^n \to \mathbb{R}$ can be written as $f = \sum_{S\subseteq\{1,\ldots,n\}} \widehat{f}(S)W_S$, where for all $x = (x_1, \ldots, x_n) \in \{-1,1\}^n$, $W_S(x) := \prod_{i\in S} x_i$ and $\widehat{f}(S) := \int_{\{-1,1\}^n} f(x)W_S(x)d\mu(x)$. For any $t \geq 0$, define $e^{-tL}f := \sum_{S\subseteq\{1,\ldots,n\}} e^{-t|S|}\widehat{f}(S)W_S$, $Lf := \sum_{S\subseteq\{1,\ldots,n\}} |S|\,\widehat{f}(S)W_S$, and $\forall\ 1 \leq p < \infty$, define $\|f\|_p := (\int_{\{-1,1\}^n} |f(x)|^p\, d\mu(x))^{1/p}$. For all $i \in \{1,\ldots,n\}$, define the **influence $I_i f$ of the i^{th} variable on f** by

$$I_i f := \sum_{S\subseteq\{1,\ldots,n\}\colon\, i\in S} (\widehat{f}(S))^2.$$

Let $k \geq 1$, $k \in \mathbb{Z}$. The Conjecture [MN14, Remark 5.5] says: $\forall\ p > 1$, $\exists\ c(p) > 0$ such that

$$\widehat{f}(S) = 0\ \ \forall\, S \subseteq \{1,\ldots,n\} \text{ with } |S| < k \quad\Longrightarrow\quad \forall\, t > 0,\quad \|e^{-tL}f\|_p \leq e^{-tkc(p)}\|f\|_p. \tag{16}$$

Equivalently, Conjecture (16) is a **"higher order" Poincaré inequality**: $\forall\ p > 1$, $\exists\ c(p) > 0$ such that

$$\widehat{f}(S) = 0\ \ \forall\, S \subseteq \{1,\ldots,n\} \text{ with } |S| < k \quad\Longrightarrow\quad \int |f|^{p-1}\operatorname{sign}(f)Lf d\mu \geq kc(p)\int |f|^p\, d\mu.$$

A weaker form of (16) with the term $e^{-tkc(p)}$ replaced by $e^{-\min(t,t^2)kc(p)}$ can be proven [MN14, Lemma 5.4] using Hölder's inequality and the **hypercontractive inequality** [Sta59, Fed69, Bon70, Nel73, Gro75, Bec75]:

$$\forall\ 1 < p < q < \infty,\quad \forall\, t > \frac{1}{2}\log\left(\frac{q-1}{p-1}\right),\qquad \|e^{-tL}f\|_q \leq \|f\|_p. \tag{17}$$

Using the hypercontractive inequality (17), Kahn, Kalai and Linial proved the following famous inequality [KKL88], resolving a conjecture of Ben-Or and Linial.

Theorem 14 (**Kahn-Kalai-Linial**)**.** [KKL88, Theorem 3.1] *There exists a universal constant $c > 0$ such that, $\forall\ f\colon \{-1,1\}^n \to \{-1,1\}$, we have* $\max_{i=1,\ldots,n} I_i f \geq c(\int[f - \int f d\mu]^2 d\mu)(\log n)/n$.

This Theorem says that a discrete function with values in $\{-1,1\}$ must have some asymmetry in its Fourier coefficients. To see this, note that it is easy to construct a function $f\colon \{-1,1\}^n \to \mathbb{R}$ such that $\max_{i=1,\ldots,n} I_i f \leq 10(\int[f - \int f d\mu]^2 d\mu)/n$, just by choosing f such that $\widehat{f}(S) = 2/(n(n-1))$ for all $|S| = 2$ and such that $\widehat{f}(S) = 0$ for all other $S \subseteq \{1,\ldots,n\}$. Note that the Fourier coefficients of f are then symmetric with respect to permutations on the set $\{1,\ldots,n\}$, but f does not take values $\{-1,1\}$, so Theorem 14 does not apply.

6.2. **Our Contribution.** In [HMO14], together with Mossel and Oleszkiewicz, we prove the case $k = 1$ of the Conjecture (16) of Mendel and Naor. In fact, we prove (16) for any probability space with a symmetric Markov semigroup P_t whose generator $L := -\frac{d}{dt}P_t|_{t=0^+}$ satisfies an L_2 Poincaré inequality. We then answer a question of Hatami and Kalai, showing that Theorem 14 cannot be strengthened unless a logarithmic number of Fourier coefficients of the function f vanish. That is,

Theorem 15 ([HMO14])**.** *There exists $c > 0$ such that, for all $n \in \mathbb{N}$, there exists $f\colon \{-1,1\}^n \to \{-1,1\}$ with $\widehat{f}(S) = 0$ for all $S \subseteq \{1,\ldots,n\}$ with $|S| \leq \log n$ such that $\max_{i=1,\ldots,n} I_i f \leq c(\log n)/n$.*

In the case that $\widehat{f}(S) = 0$ for all $S \subseteq \{1,\ldots,n\}$ with $|S| \leq C(n)\log n$, the equality $\sum_{i=1}^n I_i f = \sum_{S\subseteq\{1,\ldots,n\}} |S|\,(\widehat{f}(S))^2$ shows that $\max_{i=1,\ldots,n} I_i f \geq C(n)(\log n)/n$. So, there is a phase transition in the possible behavior of the maximum influence $\max_{i=1,\ldots,n} I_i f$, which occurs when $C(n) > 0$ is bounded or unbounded as $n \to \infty$.

Finally, we demonstrated a generalization of Talagrand's inequality for functions $f\colon \{-1,1\}^n \to \{-1,1\}$ with $\widehat{f}(S) = 0$ for all $S \subseteq \{1,\ldots,n\}$ with $|S| < k$. The usual Talagrand inequality then corresponds to the case $k = 1$.

Theorem 16 ([HMO14])**.** *Let $k \geq 1$. Let $f\colon \{-1,1\}^n \to \mathbb{R}$ with $\widehat{f}(S) = 0$ for all $S \subseteq \{1,\ldots,n\}$ with $|S| < k$. $\forall\ i = 1,\ldots,n$, let $\frac{\partial}{\partial x_i} f(x) := [f(x_1,\ldots,x_n) - f(x_1,\ldots,x_{i-1},-x_i,x_{i+1},\ldots,x_n)]/2$, where $x = (x_1,\ldots,x_n) \in \{-1,1\}^n$. Then*

$$\|f\|_2^2 \leq 6\sum_{i=1}^n \frac{\|\frac{\partial}{\partial x_i} f\|_2^2}{k + \log\left(\|\frac{\partial}{\partial x_i} f\|_2 / \|\frac{\partial}{\partial x_i} f\|_1\right)}. \tag{18}$$

6.3. **Potential Developments.** Proving the case $k > 1$ of the Conjecture (16) of Mendel and Naor would give improved understanding of analysis in non-Euclidean spaces, going beyond (or supplementing) Littlewood-Paley theory in non-doubling metric measure spaces. Also, the solution of this problem would improve our understanding of expander graphs. Recent partial progress on Conjecture 16 was announced in [EI20], though Conjecture 16 remains largely open at present.

References

[ABKK17] Naman Agarwal, Afonso S. Bandeira, Konstantinos Koiliaris, and Alexandra Kolla, *Multisection in the stochastic block model using semidefinite programming*, pp. 125–162, Springer International Publishing, Cham, 2017.

[AL18] Arash A. Amini and Elizaveta Levina, *On semidefinite relaxations for the block model*, Ann. Statist. **46** (2018), no. 1, 149–179.

[Alm76] F. J. Almgren, Jr., *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Mem. Amer. Math. Soc. **4** (1976), no. 165, viii+199. MR 0420406

[AMSE87] Yousef Alavi, Paresh J. Malde, Allen J. Schwenk, and Paul Erdős, *The vertex independence sequence of a graph is not constrained*, vol. 58, 1987, Eighteenth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, Fla., 1987), pp. 15–23. MR 944684

[AN06] Noga Alon and Assaf Naor, *Approximating the cut-norm via Grothendieck's inequality*, SIAM J. Comput. **35** (2006), no. 4, 787–803 (electronic). MR 2203567 (2006k:68176)

[AS15] Emmanuel Abbe and Colin Sandon, *Community detection in general stochastic block models: Fundamental limits and efficient algorithms for recovery*, Proceedings of the 2015 IEEE 56th Annual Symposium on Foundations of Computer Science (FOCS) (Washington, DC, USA), FOCS '15, IEEE Computer Society, 2015, pp. 670–688.

[Ban65] John F. Banzhaf, *Weighted Voting Doesn't Work: A Mathematical Analysis*, Rutgers Law Rev. **19** (1965), 317–343.

[Bar01] Franck Barthe, *An isoperimetric result for the Gaussian measure and unconditional sets*, Bulletin of the London Mathematical Society **33** (2001), 408–416.

[BBJ17] Marco Barchiesi, Alessio Brancolini, and Vesa Julin, *Sharp dimension free quantitative estimates for the Gaussian isoperimetric inequality*, Ann. Probab. **45** (2017), no. 2, 668–697. MR 3630285

[BBO14] Patrick Bahls, Elizabeth Bailey, and McCabe Olsen, *New families of graphs whose independence polynomials have only real zeros*, Australas. J. Combin. **60** (2014), 128–135. MR 3251932

[BdC84] João Lucas Barbosa and Manfredo do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z. **185** (1984), no. 3, 339–353. MR 731682 (85k:58021c)

[Bec75] William Beckner, *Inequalities in Fourier analysis*, Ann. of Math. (2) **102** (1975), no. 1, 159–182. MR 0385456 (52 #6317)

[BG20] Abdul Basit and David Galvin, *On the independent set sequence of a tree*, Preprint, arXiv:2006.12562, 2020.

[BJ20] Marco Barchiesi and Vesa Julin, *Symmetry of minimizers of a Gaussian isoperimetric problem*, Probab. Theory Related Fields **177** (2020), no. 1-2, 217–256. MR 4095016

[BL96] D. Bakry and M. Ledoux, *Lévy-Gromov's isoperimetric inequality for an infinite-dimensional diffusion generator*, Invent. Math. **123** (1996), no. 2, 259–281. MR 1374200

[BMMN13] Mark Braverman, Konstantin Makarychev, Yury Makarychev, and Assaf Naor, *The Grothendieck constant is strictly smaller than Krivine's bound*, Forum Math. Pi **1** (2013), e4, 42. MR 3141414

[Bob97] S. G. Bobkov, *An isoperimetric inequality on the discrete cube, and an elementary proof of the isoperimetric inequality in gauss space*, Ann. Probab. **25** (1997), no. 1, 206–214.

[Bon70] Aline Bonami, *Étude des coefficients de Fourier des fonctions de $L^p(G)$*, Ann. Inst. Fourier (Grenoble) **20** (1970), no. fasc. 2, 335–402 (1971). MR 0283496 (44 #727)

[Bor75] Christer Borell, *The Brunn-Minkowski inequality in Gauss space*, Invent. Math. **30** (1975), no. 2, 207–216. MR 0399402 (53 #3246)

[Bor85] ———, *Geometric bounds on the Ornstein-Uhlenbeck velocity process*, Z. Wahrsch. Verw. Gebiete **70** (1985), no. 1, 1–13. MR 795785 (87k:60103)

[Bor03] ———, *The Ehrhard inequality*, C. R. Math. Acad. Sci. Paris **337** (2003), no. 10, 663–666. MR 2030108 (2004k:60102)

[BPT13] DP Bourne, MA Peletier, and F Theil, *Optimality of the triangular lattice for a particle system with wasserstein interaction*, To appear in Communications in Mathematical Physics (2013).

[BR15] Jonah Brown-Cohen and Prasad Raghavendra, *Combinatorial optimization algorithms via polymorphisms*, Electronic Colloquium on Computational Complexity (ECCC) **22** (2015), 7.

[BS01] A. Burchard and M. Schmuckenschlger, *Comparison theorems for exit times*, Geometric & Functional Analysis GAFA **11** (2001), no. 4, 651–692 (English).

[CCH+08] J. Corneli, I. Corwin, S. Hurder, V. Sesum, Y. Xu, E. Adams, D. Davis, M. Lee, R. Visocchi, and N. Hoffman, *Double bubbles in Gauss space and spheres*, Houston J. Math. **34** (2008), no. 1, 181–204. MR 2383703 (2009e:53007)

[CE15] Amin Coja-Oghlan and Charilaos Efthymiou, *On independent sets in random graphs*, Random Struct. Algorithms **47** (2015), no. 3, 436–486.

[Cha06] Sourav Chatterjee, *A generalization of the Lindeberg principle*, Ann. Probab. **34** (2006), no. 6, 2061–2076. MR 2294976 (2008c:60028)

[CIMW13] Tobias Holck Colding, Tom Ilmanen, William P. Minicozzi, II, and Brian White, *The round sphere minimizes entropy among closed self-shrinkers*, J. Differential Geom. **95** (2013), no. 1, 53–69. MR 3128979

[CM12] Tobias H. Colding and William P. Minicozzi, II, *Generic mean curvature flow I: generic singularities*, Ann. of Math. (2) **175** (2012), no. 2, 755–833. MR 2993752

[CR11] Amit Chakrabarti and Oded Regev, *An optimal lower bound on the communication complexity of gap Hamming distance*, Proc. 43rd Annual ACM Symposium on the Theory of Computing, 2011, pp. 51–60.

[CS07] Rustum Choksi and Peter Sternberg, *On the first and second variations of a nonlocal isoperimetric problem*, J. Reine Angew. Math. **611** (2007), 75–108. MR 2360604 (2008j:49062)

[DR18] Sara Daneri and Eris Runa, *Exact periodic stripes for minimizers of a local/nonlocal interaction functional in general dimension*, Archive for Rational Mechanics and Analysis (2018).

[DSS16] Jian Ding, Allan Sly, and Nike Sun, *Maximum independent sets on random regular graphs*, Acta Math. **217** (2016), no. 2, 263–340.

[EH89] Klaus Ecker and Gerhard Huisken, *Mean curvature evolution of entire graphs*, Ann. of Math. (2) **130** (1989), no. 3, 453–471. MR 1025164

[EI20] Alexandros Eskenazis and Paata Ivanisvili, *Dimension independent Bernstein–Markov inequalities in Gauss space*, J. Approx. Theory **253** (2020), 105377. MR 4064847

[Eld15] Ronen Eldan, *A two-sided estimate for the gaussian noise stability deficit*, Inventiones mathematicae **201** (2015), no. 2, 561–624 (English).

[Eva93] Lawrence C. Evans, *Convergence of an algorithm for mean curvature motion*, Indiana Univ. Math. J. **42** (1993), no. 2, 533–557. MR 1237058 (95d:58023)

[Fed69] Paul Federbush, *Partially alternate derivation of a result of nelson*, Journal of Mathematical Physics **10** (1969), no. 1, 50–52.

[FGRW12] Vitaly Feldman, Venkatesan Guruswami, Prasad Raghavendra, and Yi Wu, *Agnostic learning of monomials by halfspaces is hard*, SIAM J. Comput. **41** (2012), no. 6, 1558–1590. MR 3029261

[Fri90] A. M. Frieze, *On the independence number of random graphs*, Discrete Math. **81** (1990), no. 2, 171–175.
[Gal11] David Galvin, *Two problems on independent sets in graphs*, Discrete Math. **311** (2011), no. 20, 2105–2112. MR 2825652
[Gal12] ______, *The independent set sequence of regular bipartite graphs*, Discrete Math. **312** (2012), no. 19, 2881–2892. MR 2946949
[GH18] David Galvin and Justin Hilyard, *The independent set sequence of some families of trees*, Australas. J. Combin. **70** (2018), 236–252. MR 3737109
[GM12] Alessandro Giuliani and Stefan Müller, *Striped periodic minimizers of a two-dimensional model for martensitic phase transitions*, Communications in Mathematical Physics **309** (2012), no. 2, 313–339.
[Gro53] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. São Paulo **8** (1953), 1–79. MR 0094682 (20 #1194)
[Gro75] Leonard Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), no. 4, 1061–1083. MR 0420249 (54 #8263)
[Gro83] Mikhael Gromov, *Filling Riemannian manifolds*, J. Differential Geom. **18** (1983), no. 1, 1–147. MR 697984
[GS16] Alessandro Giuliani and Robert Seiringer, *Periodic striped ground states in ising models with competing interactions*, Communications in Mathematical Physics **347** (2016), no. 3, 983–1007.
[GW95] Michel X. Goemans and David P. Williamson, *Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming*, J. Assoc. Comput. Mach. **42** (1995), no. 6, 1115–1145. MR 1412228 (97g:90108)
[Hei14] Steven Heilman, *Euclidean partitions optimizing noise stability*, Electron. J. Probab. **19** (2014), no. 71, 37. MR 3256871
[Hei15] ______, *Low correlation noise stability of symmetric sets*, to appear, Journal of Theoretical Probability. Preprint, arXiv:1511.00382, 2015.
[Hei17] ______, *Symmetric convex sets with minimal Gaussian surface area*, to appear, American Journal of Mathematics. Preprint, arXiv:1705.06643., 2017.
[Hei18] ______, *The structure of Gaussian minimal bubbles*, to appear, Journal of Geometric Analysis. Preprint, arXiv:1805.10203, 2018.
[Hei19] ______, *Stable Gaussian minimal bubbles*, Preprint, arXiv:1901.03934, 2019.
[Hei20a] Steven Heilman, *Designing stable elections: A survey*, Preprint, arXiv:2006.05460, 2020.
[Hei20b] ______, *Independent sets of random trees and of sparse random graphs*, Preprint, arXiv:2006.04756, 2020.
[Hei20c] Steven Heilman, *A periodic isoperimetric problem related to the unique games conjecture*, Random Structures Algorithms **56** (2020), no. 1, 154–168. MR 4052850
[Hei20d] Steven Heilman, *Tree/endofunction bijections and concentration inequalities*, Preprint, arXiv:2006.06724, 2020.
[HJN13] Steven Heilman, Aukosh Jagannath, and Assaf Naor, *Solution of the propeller conjecture in* $\mathbb{R}^3$, Discrete & Computational Geometry **50** (2013), no. 2, 263–305 (English).
[HLW06] Shlomo Hoory, Nathan Linial, and Avi Wigderson, *Expander graphs and their applications*, Bull. Amer. Math. Soc. (N.S.) **43** (2006), no. 4, 439–561 (electronic). MR 2247919 (2007h:68055)
[HMN16] Steven Heilman, Elchanan Mossel, and Joe Neeman, *Standard simplices and pluralities are not the most noise stable*, Israel Journal of Mathematics **213** (2016), no. 1, 33–53.
[HMO14] Steven Heilman, Elchanan Mossel, and Krzysztof Oleszkiewicz, *Strong contraction and influences in tail spaces*, to appear, Transactions of the AMS, 2014.
[HMRR02] Michael Hutchings, Frank Morgan, Manuel Ritoré, and Antonio Ros, *Proof of the double bubble conjecture*, Ann. of Math. (2) **155** (2002), no. 2, 459–489. MR 1906593 (2003c:53013)
[Hur02] A. Hurwitz, *Sur quelques applications géométriques des séries de Fourier*, Ann. Sci. École Norm. Sup. (3) **19** (1902), 357–408. MR 1509016
[Hut97] Michael Hutchings, *The structure of area-minimizing double bubbles*, J. Geom. Anal. **7** (1997), no. 2, 285–304. MR 1646776
[HV16] Steven Heilman and Thomas Vidick, *A moment majorization principle for random matrix ensembles with applications to hardness of the noncommutative Grothendieck problem*, preprint, arXiv:1603.05620, 2016.
[HWX16] Bruce Hajek, Yihong Wu, and Jiaming Xu, *Achieving exact cluster recovery threshold via semidefinite programming*, IEEE Trans. Inf. Theor. **62** (2016), no. 5, 2788–2797.
[IM12] Marcus Isaksson and Elchanan Mossel, *Maximally stable Gaussian partitions with discrete applications*, Israel J. Math. **189** (2012), 347–396. MR 2931402
[JSV04] Mark Jerrum, Alistair Sinclair, and Eric Vigoda, *A polynomial-time approximation algorithm for the permanent of a matrix with nonnegative entries*, J. ACM **51** (2004), no. 4, 671–697.
[Kal02] Gil Kalai, *A fourier-theoretic perspective on the condorcet paradox and arrow's theorem*, Advances in Applied Mathematics **29** (2002), no. 3, 412 – 426.

[Kan14] Daniel M. Kane, *The correct exponent for the Gotsman-Linial conjecture*, Comput. Complexity **23** (2014), no. 2, 151–175. MR 3212596

[Kho02] Subhash Khot, *On the power of unique 2-prover 1-round games*, Proceedings of the Thirty-Fourth Annual ACM Symposium on Theory of Computing (New York), ACM, 2002, pp. 767–775 (electronic). MR MR2121525

[KKL88] Jeff Kahn, Gil Kalai, and Nathan Linial, *The influence of variables on boolean functions*, Proc. of 29th Annual IEEE Symposium on Foundations of Computer Science, 1988, pp. 68–80.

[KKMO07] Subhash Khot, Guy Kindler, Elchanan Mossel, and Ryan O'Donnell, *Optimal inapproximability results for MAX-CUT and other 2-variable CSPs?*, SIAM J. Comput. **37** (2007), no. 1, 319–357. MR 2306295 (2008d:68035)

[KM16] Subhash Khot and Dana Moshkovitz, *Candidate hard unique game*, Proceedings of the Forty-eighth Annual ACM Symposium on Theory of Computing, STOC '16, ACM, 2016.

[KMS18] Subhash Khot, Dor Minzer, and Muli Safra, *Pseudorandom sets in grassmann graph have near-perfect expansion*, Electronic Colloquium on Computational Complexity (ECCC) **25** (2018), 6.

[KN09] Subhash Khot and Assaf Naor, *Approximate kernel clustering*, Mathematika **55** (2009), no. 1-2, 129–165. MR 2573605 (2011c:68166)

[KN13] ———, *Sharp kernel clustering algorithms and their associated grothendieck inequalities*, Random Structures & Algorithms **42** (2013), no. 3, 269–300.

[Law70] H. Blaine Lawson, Jr., *Complete minimal surfaces in S^3*, Ann. of Math. (2) **92** (1970), 335–374. MR 0270280

[Led94] Michel Ledoux, *Semigroup proofs of the isoperimetric inequality in Euclidean and Gauss space*, Bull. Sci. Math. **118** (1994), no. 6, 485–510. MR 1309086 (96c:49061)

[Led96] ———, *Isoperimetry and Gaussian analysis*, Lectures on probability theory and statistics (Saint-Flour, 1994), Lecture Notes in Math., vol. 1648, Springer, Berlin, 1996, pp. 165–294. MR 1600888 (99h:60002)

[Lév51] Paul Lévy, *Problèmes concrets d'analyse fonctionnelle. Avec un complément sur les fonctionnelles analytiques par F. Pellegrino*, Gauthier-Villars, Paris, 1951, 2d ed. MR 0041346 (12,834a)

[LM02] Vadim E. Levit and Eugen Mandrescu, *On well-covered trees with unimodal independence polynomials*, Proceedings of the Thirty-third Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 2002), vol. 159, 2002, pp. 193–202. MR 1985180

[LM03] ———, *On unimodality of independence polynomials of some well-covered trees*, Discrete mathematics and theoretical computer science, Lecture Notes in Comput. Sci., vol. 2731, Springer, Berlin, 2003, pp. 237–256. MR 2062222

[LM07] ———, *Independence polynomials and the unimodality conjecture for very well-covered, quasi-regularizable, and perfect graphs*, Graph theory in Paris, Trends Math., Birkhäuser, Basel, 2007, pp. 243–254. MR 2279180

[LP68] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in L_p-spaces and their applications*, Studia Math. **29** (1968), 275–326. MR 0231188 (37 #6743)

[Min96] Hermann Minkowski, *Geometrie der Zahlen*, 1896.

[MN14] Manor Mendel and Assaf Naor, *Nonlinear spectral calculus and super-expanders*, Publ. Math. Inst. Hautes Études Sci. **119** (2014), 1–95. MR 3210176

[MN15a] Elchanan Mossel and Joe Neeman, *Robust dimension free isoperimetry in gaussian space*, The Annals of Probability **43** (2015), no. 3, 971–991.

[MN15b] ———, *Robust optimality of Gaussian noise stability*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 2, 433–482. MR 3317748

[MN18a] Emanuel Milman and Joe Neeman, *The Gaussian double-bubble conjecture*, Preprint, arXiv:1801.09296, 2018.

[MN18b] ———, *The Gaussian multi-bubble conjecture*, Preprint, arXiv:1805.10961, 2018.

[MOO10] Elchanan Mossel, Ryan O'Donnell, and Krzysztof Oleszkiewicz, *Noise stability of functions with low influences: invariance and optimality*, Ann. of Math. (2) **171** (2010), no. 1, 295–341. MR 2630040 (2012a:60091)

[Mos10] Elchanan Mossel, *Gaussian bounds for noise correlation of functions*, Geom. Funct. Anal. **19** (2010), no. 6, 1713–1756. MR 2594620 (2011b:60080)

[Mos12] ———, *A quantitative arrow theorem*, Probability Theory and Related Fields **154** (2012), no. 1, 49–88.

[MP14] Shahar Mendelson and Grigoris Paouris, *On the singular values of random matrices*, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 4, 823–834. MR 3191978

[MPW16] Ankur Moitra, William Perry, and Alexander S. Wein, *How robust are reconstruction thresholds for community detection?*, STOC'16—Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing, ACM, New York, 2016, pp. 828–841. MR 3536617

[MR15] Matthew McGonagle and John Ross, *The hyperplane is the only stable, smooth solution to the isoperimetric problem in Gaussian space*, Geom. Dedicata **178** (2015), 277–296. MR 3397495
[MS02] Rolando Magnanini and Shigeru Sakaguchi, *Matzoh ball soup: heat conductors with a stationary isothermic surface*, Ann. of Math. (2) **156** (2002), no. 3, 931–946. MR 1954240
[Nel73] Edward Nelson, *The free Markoff field*, J. Functional Analysis **12** (1973), 211–227. MR 0343816 (49 #8556)
[NPR10] Ivan Nourdin, Giovanni Peccati, and Gesine Reinert, *Invariance principles for homogeneous sums: universality of Gaussian Wiener chaos*, Ann. Probab. **38** (2010), no. 5, 1947–1985. MR 2722791 (2011g:60043)
[NR14] Assaf Naor and Oded Regev, *Krivine schemes are optimal*, Proc. Amer. Math. Soc. **142** (2014), no. 12, 4315–4320. MR 3266999
[O'D] Ryan O'Donnell, *Social choice, computational complexity, gaussian geometry, and boolean functions*, In proceedings of the 2014 ICM.
[O'D12] ———, *Open problems in analysis of boolean functions*, Preprint, arXiv:1204.6447v1, 2012.
[O'D14] ———, *Analysis of Boolean functions*, Cambridge University Press, 2014.
[Pis12] Gilles Pisier, *Grothendieck's theorem, past and present*, Bull. Amer. Math. Soc. (N.S.) **49** (2012), no. 2, 237–323. MR 2888168
[Pla73] Joseph Antoine Ferdinand Plateau, *Statique expérimentale et théorique des liquides soumis aux seules forces moleculaires*.
[Rei08] BenW. Reichardt, *Proof of the double bubble conjecture in* $\mathbb{R}^n$, Journal of Geometric Analysis **18** (2008), no. 1, 172–191 (English).
[Ros01] Antonio Ros, *The isoperimetric problem*, Proceedings from the Clay Mathematics Institute Summer School, MSRI, Berkeley, CA, 2001, Available at http://www.ugr.es/∼aros/isoper.pdf.
[Rot79] V. I. Rotar′, *Limit theorems for polylinear forms*, J. Multivariate Anal. **9** (1979), no. 4, 511–530. MR 556909 (81m:60039)
[RS09] Prasad Raghavendra and David Steurer, *Towards computing the Grothendieck constant*, Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms (Philadelphia, PA), SIAM, 2009, pp. 525–534. MR 2809257 (2012i:90109)
[SC74] V. N. Sudakov and B. S. Cirel′son, *Extremal properties of half-spaces for spherically invariant measures*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **41** (1974), 14–24, 165, Problems in the theory of probability distributions, II. MR 0365680 (51 #1932)
[Sch72] H. A. Schwarz, *Gesammelte mathematische Abhandlungen. Band I, II*, Chelsea Publishing Co., Bronx, N.Y., 1972, Nachdruck in einem Band der Auflage von 1890. MR 0392470
[Sim68] James Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2) **88** (1968), 62–105. MR 0233295
[Sim83] Leon Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. (2) **118** (1983), no. 3, 525–571. MR 727703
[SM96] John M. Sullivan and Frank Morgan, *Open problems in soap bubble geometry*, Internat. J. Math. **7** (1996), no. 6, 833–842. MR 1417788
[SS54] L. S. Shapley and Martin Shubik, *A method for evaluating the distribution of power in a committee system.*, American Political Science Review **48** (1954), 787–792.
[Sta59] A. J. Stam, *Some inequalities satisfied by the quantities of information of Fisher and Shannon*, Information and Control **2** (1959), 101–112. MR 0109101 (21 #7813)
[Ste38] Jakob Steiner, *Einfache beweise der isoeperimetrische hauptsätze*, J. Reine Angew. Math. **18** (1838), 281–296.
[Tal95] Michel Talagrand, *Concentration of measure and isoperimetric inequalities in product spaces*, Inst. Hautes Études Sci. Publ. Math. (1995), no. 81, 73–205. MR 1361756
[Tay76] Jean E. Taylor, *The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces*, Ann. of Math. (2) **103** (1976), no. 3, 489–539. MR 0428181
[The06] Florian Theil, *A proof of crystallization in two dimensions*, Communications in Mathematical Physics **262** (2006), no. 1, 209–236.
[Wei27] Karl Weierstrass, *Mathematische Werke. VII. Vorlesungen über Variationsrechnung*, 1927.
[WZ11] Yi Wang and Bao-Xuan Zhu, *On the unimodality of independence polynomials of some graphs*, European J. Combin. **32** (2011), no. 1, 10–20. MR 2727455
[Zhu07] Zhi-Feng Zhu, *The unimodality of independence polynomials of some graphs*, Australas. J. Combin. **38** (2007), 27–33. MR 2324272
[Zhu16] Bao-Xuan Zhu, *Clique cover products and unimodality of independence polynomials*, Discrete Appl. Math. **206** (2016), 172–180. MR 3490439
[Zhu20] Jonathan J. Zhu, *On the entropy of closed hypersurfaces and singular self-shrinkers*, J. Differential Geom. **114** (2020), no. 3, 551–593. MR 4072205