

A MOMENT MAJORIZATION PRINCIPLE FOR RANDOM MATRIX ENSEMBLES WITH APPLICATIONS TO HARDNESS OF THE NONCOMMUTATIVE GROTHENDIECK PROBLEM

STEVEN HEILMAN AND THOMAS VIDICK

ABSTRACT. We prove a moment majorization principle for matrix-valued functions with domain $\{-1, 1\}^m$, $m \in \mathbb{N}$. The principle is an inequality between higher-order moments of a non-commutative multilinear polynomial with different random matrix ensemble inputs, where each variable has small influence and the variables are instantiated independently.

This technical result can be interpreted as a noncommutative generalization of one of the two inequalities of the seminal invariance principle of Mossel, O’Donnell and Oleszkiewicz. Our main application is sharp Unique Games hardness for two versions of the noncommutative Grothendieck inequality. This generalizes a result of Raghavendra and Steurer who established hardness of approximation for the commutative Grothendieck inequality. A similar application was proven recently by Briët, Regev and Saket using different techniques.

1. INTRODUCTION

1.1. A noncommutative moment majorization theorem. We study matrix-valued functions f with domain $\{-1, 1\}^m$ within the context of probability theory and Fourier analysis. More specifically, we study functions f such that, for every $\sigma \in \{-1, 1\}^m$, the operator norm of $f(\sigma)$ is at most 1. The special case when f is valued in the two-point space $\{-1, 1\}$ has been studied extensively within theoretical computer science [KKL88], but also in diverse areas such as combinatorics, isoperimetry [Tal94], or social choice theory [Kal02, MOO10, MN15]. (For a more comprehensive list of references and discussion, see e.g. the survey [O’D14b].) In applications to theoretical computer science, a function $f: \{-1, 1\}^m \rightarrow \{-1, 1\}$ can be used to represent an instance of a combinatorial optimization problem. That is, the function f can be thought of as a list of elements of $\{-1, 1\}$, seen as a Boolean assignment to the 2^m variables of some constraint satisfaction problem. Functions with domain $\{-1, 1\}^m$ and range the simplex $\{(x_1, \dots, x_n) \in \mathbb{R}^n: \sum_{i=1}^n x_i = 1, x_1 \geq 0, \dots, x_n \geq 0\}$ have also been considered [KN09, KN13, IM12]. Projecting f onto each coordinate gives a family of functions with range $[0, 1]$, so that similar tools to the Boolean case can be applied. Here our main application, and motivation, is to the noncommutative Grothendieck inequality (NCGI), an inequality which involves two orthogonal (in the real case) or unitary (in the complex case) matrix variables of fixed dimension. This setting leads us to consider *matrix-valued functions* with domain $\{-1, 1\}^m$ (considering $m > 1$ will allow us to “combine” multiple instances of NCGI acting on partially overlapping sets of variables).

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In many of the applications listed above a standard manipulation is to extend f to a multilinear polynomial, so that the distribution of f can be studied under different distributions on its domain, such as the standard Gaussian distribution. In our setting it is natural (and, as we will see, for our purposes necessary) to investigate the behavior of matrix-valued functions under distributions on their domain that allow the possibility for matrix variables. For any set S , let $M_n(S)$ denotes the $n \times n$ matrices with entries in S . Any $f : \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$ can be extended to a multilinear polynomial in m *noncommutative* variables with matrix coefficients. Consider for instance the case $m = 2$ and the polynomial $f(\sigma_1, \sigma_2) = \sigma_1 \sigma_2$, where $\sigma_1, \sigma_2 \in \{-1, 1\}$. Since the variables σ_1, σ_2 commute, it is not necessary to specify the order in which the product of the variables is taken in f . However, once f is extended to matrix variables X_1, X_2 , an ordering needs to be specified. We adopt the convention of ordering matrix variables by increasing order, e.g. $f(X_1, X_2) = X_1 X_2$.

Let d, m, n be positive integers. For us, a noncommutative multilinear polynomial of degree d in m variables can be expressed as

$$Q(X_1, \dots, X_m) = \sum_{S \subseteq \{1, \dots, m\}: |S| \leq d} \widehat{Q}(S) \prod_{i \in S} X_i,$$

where $\widehat{Q}(S)$ is an $n \times n$ complex matrix for every $S \subseteq \{1, \dots, m\}$, X_1, \dots, X_m are non-commutative $n \times n$ matrix variables, and the product $\prod_{i \in S} X_i$ is always taken in increasing order. For example, $\prod_{i \in \{1, 2\}} X_i = X_1 X_2$. Noncommutative polynomials appear in many other contexts, most notably, within free probability [Voi91, Theorem 3.3]. In addition there is a general theory of so-called nc-functions [KVV14], but this theory does not seem to apply to the noncommutative polynomials we consider here. (An nc function h is a function defined on matrices of any dimension, such that, for any $n \geq 1$, and for any $n \times n$ matrices A, B, C such that C is invertible, $h(CAC^{-1}) = Ch(A)C^{-1}$ and $h(A \oplus B) = h(A) \oplus h(B)$. Neither property is satisfied by a general matrix-valued non-commutative polynomial as defined below.)

Our main goal consists in bounding the moments of Q for different random matrix distributions in the domain, when all partial derivatives of Q are small (i.e. when Q has small influences). In particular, we would like to say that the moments of polynomials Q with small influences under Gaussian random matrix inputs are close to the moments of Q under uniform $\{-1, 1\}^m$ inputs. Unfortunately, this task is in general impossible. For example, consider the linear polynomial $Q(X_1, \dots, X_m) = (X_1 + \dots + X_m)/\sqrt{m}$. For any square matrix A , let $|A| = (AA^*)^{1/2}$. Let b_1, \dots, b_m be i.i.d. uniform random variables in $\{-1, 1\}$, and let I denote the $n \times n$ identity matrix. Then $\mathbb{E}(1/n) \text{Tr} |Q(b_1 I, \dots, b_m I)|^4 = 3 - 2/m$. On the other hand, let G_1, \dots, G_m be $n \times n$ independent Wigner matrices with real Gaussian entries. In this case $Q(G_1, \dots, G_m)$ is equal in distribution to G_1 , and in particular $\lim_{n \rightarrow \infty} (1/n) \text{Tr} |Q(G_1, \dots, G_m)|^4 = 2$, by the semicircle law. Thus even though the first and second moments of the input distributions match, i.e. $\mathbb{E}b_1 = 0$, $\mathbb{E}b_1^2 = 1$, $\mathbb{E}G_1 = 0$ and $\mathbb{E}G_1 G_1^* = I$, the associated moments of Q can be very different.

In summary, a general invariance principle cannot hold in this noncommutative setting. We could instead try to prove a weaker statement such as: the moments of Q under non-commutative inputs with $\mathbb{E}G_1 = 0$ and $\mathbb{E}G_1 G_1^* = I$ are bounded by the moments of Q under Boolean inputs. We call such a statement a *moment majorization theorem*. Unfortunately, this is also not true in general, as we now show. For any $1 \leq i \leq n$, let A_i be the $n \times n$

matrix with a 1 in the first row and i^{th} column, and zeros in all other entries. Then for any $1 \leq j, k \leq n$ with $j \neq k$, we have $A_j A_k^* = 0$. Consider the linear polynomial

$$Q(X_1, \dots, X_n) = \sum_{i=0}^{n-1} B^i A X_i,$$

which is such that $\mathbb{E} \text{Tr} |Q(b_1, \dots, b_n)|^4 = n$. Now let H_1 be a uniformly random Haar distributed $n \times n$ unitary matrix, and let H_2, \dots, H_n be independent copies of H_1 . Then

$$\mathbb{E} \text{Tr} |Q(H_1, \dots, H_n)|^4 = n \text{Tr} |A_1|^4 + n(n-1) \mathbb{E} \text{Tr} (|A_1|^2 H_1 |A_1|^2 H_1^*) = n + n(n-1)/n = 2n - 1.$$

Since a general noncommutative majorization principle cannot hold we instead establish a limited moment majorization theorem, which will nevertheless be sufficient for our applications. We make two changes. We first increase the dimension of the random matrix inputs G_1, \dots, G_m . That is, we allow the variables of Q to take values in the set of $p \times p$ matrices with $p > n$ by defining the $p \times p$ matrix

$$\iota(A) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

for any $n \times n$ matrix A , and

$$Q^\iota(X_1, \dots, X_m) = \sum_{S \subseteq \{1, \dots, m\}} \iota(\widehat{Q}(S)) \prod_{i \in S} X_i,$$

where X_1, \dots, X_m are noncommutative $p \times p$ matrix variables. Second, we randomly rotate G_1, \dots, G_m by $p \times p$ Haar-distributed random unitary matrices H_1, \dots, H_m .

We state one particular variant of our noncommutative moment majorization theorem. Let n be a positive integer and let p be a positive multiple of n . We write $H \sim \mathcal{H}$ to denote a $p \times p$ Haar-distributed random unitary matrix, $b \sim \mathcal{B}$ for a uniformly random $b \in \{-1, 1\}$, and $G \sim \mathcal{G}$ for any random variable taking values in $M_n(\mathbb{C})$ such that $\mathbb{E}G = 0$ and $\mathbb{E}GG^* = I$. We also write $G' \sim \mathcal{G} \otimes I$ to denote $G' = G \otimes I$ with $G \sim \mathcal{G}$, so that G' is a $p \times p$ matrix. We use the succinct notation $G_i \sim \mathcal{G}$ to denote a collection G_1, \dots, G_m of independent random matrices with distribution \mathcal{G} , and denote $Q(G_1, \dots, G_m)$ as $Q\{G_i\}$. The operator norm of a matrix A is denoted $\|A\|$.

Theorem 1.1 (Noncommutative Fourth Moment Majorization). *Let Q be a noncommutative multilinear polynomial of degree d in m variables such that $\|Q(\sigma)\| \leq 1$ for all $\sigma \in \{-1, 1\}^m$. Let $\tau := \max_{i=1, \dots, m} \sum_{S \subseteq \{1, \dots, m\}: i \in S} \text{Tr}(\widehat{Q}(S)\widehat{Q}(S)^*)$ be the maximum influence of Q . Let $G_i \sim \mathcal{G}$ and $c_2 \geq 1$ such that $\|\mathbb{E}(G_1 G_1^*)^2\| \leq c_2$. Then*

$$\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} |Q^\iota\{G_i H_i\}|^4 \leq \mathbb{E}_{b_i \sim \mathcal{B}} \frac{1}{n} \text{Tr} |Q\{b_i\}|^4 + 8(8c_2)^{4d} n^4 \tau^{1/4} + O_{m,n}(p^{-1/2}). \quad (1)$$

Remark 1.2. This Theorem is a special case of Theorem 3.9 below. The term $8(8c_2)^{4d} n^4 \tau^{1/4}$ on the right-hand side of (1) can be replaced by $(8c_2)^{4d} \tau$ by additionally assuming that $\mathbb{E}a_1 a_2 a_3 = 0$ for any a_1, a_2, a_3 which are (possibly repeated) entries of G_1 . We omit the proof of this strengthened statement, since the details are essentially identical to the proof of Theorem 3.5.

Remark 1.3. Note that, although Q^t takes values in the set of $p \times p$ matrices, the trace is normalized by $1/n$, and Q^t still “acts like” an $n \times n$ matrix. In particular the moments of Q^t do not become arbitrarily small in general; for instance it holds (see Lemma 2.6 below) that

$$\mathbb{E}_{\substack{G_i \sim \mathcal{G}^{\otimes I} \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} |Q^t \{G_i H_i\}|^2 = \mathbb{E}_{b_i \sim \mathcal{B}} \frac{1}{n} \text{Tr} |Q \{b_i\}|^2.$$

Theorem 1.1 shows that the fourth moment of Q with appropriate random matrix inputs is bounded by the fourth moment of Q with Boolean inputs. We provide a variant of this majorization principle for higher order moments (Theorem 3.9) and for increasing test functions (Theorem 3.5). These majorization principles all have dependence on the degree of the polynomial, but we also give degree-independent bounds for polynomials whose higher order coefficients decay at an exponential rate (Corollary 3.8).

Although majorization principles such as Theorem 1.1 involve the trace norm of a polynomial, we can obtain bounds on the operator norm of Q in the following way. Let $t \in \mathbb{R}$ and consider the function $t \mapsto (\max(0, |t| - 1))^2$. This function can be applied to self-adjoint matrices via spectral calculus, and $A \mapsto (\max(0, (AA^*)^{1/2} - 1))^2 = 0$ if the singular values of A are all bounded by 1. In particular, if $\|Q(\sigma)\| \leq 1$ for all $\sigma \in \{-1, 1\}^m$ then $(\max(0, (Q(\sigma)Q(\sigma)^*)^{1/2} - 1))^2 = 0$ for all $\sigma \in \{-1, 1\}^m$. The following majorization principle gives control on the operator norm of Q when we substitute appropriate random matrices into the domain of Q .

Theorem 1.4 (Noncommutative Operator Norm Majorization). *Let Q be a noncommutative multilinear polynomial of degree d . Suppose $\|Q(\sigma)\| \leq 1$ for all $\sigma \in \{-1, 1\}^m$. Let $\tau := \max_{i=1, \dots, m} \sum_{S \subseteq \{1, \dots, m\}: i \in S} \text{Tr}(\widehat{Q}(S)\widehat{Q}(S)^*)$ be the maximum influence of Q . Let $G_i \sim \mathcal{G}$ and $c_2, c_3 \geq 1$ such that $\|\mathbb{E}(G_1 G_1^*)^2\| \leq c_2$ and $\|\mathbb{E}(G_1 G_1^*)^3\| \leq c_3$. Then*

$$\mathbb{E}_{\substack{G_i \sim \mathcal{G}^{\otimes I} \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} (\max(0, |Q^t \{G_i H_i\}| - 1))^2 \leq (8c_2 c_3)^{9d} n^{1/2} \tau^{1/6} + O_{m,n}(\tau^{-1/3} p^{-1/2}). \quad (2)$$

This Theorem appears below in Theorem 3.7. We will use this theorem to argue that under the proper normalization condition a low-influence polynomial typically maps random Gaussian matrix inputs to matrices of norm not much larger than 1. The ensemble that will be of most interest for us is the following.

Example 1.5. Let $N > 0$. Let V_1, \dots, V_N be $n \times n$ complex matrices such that $\sum_{i=1}^N V_i V_i^* = 1$. Let g_1, \dots, g_N be i.i.d. standard complex Gaussian random variables (so that $\mathbb{E}g_1 = 0$ and $\mathbb{E}|g_1|^2 = 1$.) Define $G_1 = \sum_{i=1}^N g_i V_i$, and let G_2, \dots, G_m be independent copies of G_1 . Then it follows from [HT99, Corollary 2.8] that the random matrices G_1, \dots, G_m satisfy the hypothesis of Theorem 1.1 with $c_2 = 2$. In the case that V_1, \dots, V_N are real matrices and g_1, \dots, g_N are i.i.d. standard real Gaussians, the hypothesis of Theorem 1.1 is satisfied with $c_2 = 4 \cdot 2 = 8$, as follows from the complex case and the inequality $\|\mathbb{E}(\Re(G_1)\Re(G_1)^*)^2\| \leq \|\mathbb{E}(G_1 G_1^*)^2\| \leq 2$ when g_1, \dots, g_N are complex.

Here are some other examples of random matrix ensembles satisfying the hypothesis of Theorem 1.4.

Example 1.6. Example 1.5 specifically applies to Gaussian Wigner matrices as follows. Let U_1, \dots, U_{n^2} be $n \times n$ matrices such that these matrices are the standard orthonormal basis

of \mathbb{C}^{n^2} . Then G_1 is a Wigner matrix and the hypothesis of Theorem 1.1 is satisfied with $c_2 = 2$. Similarly, let $U_1, \dots, U_{n(n+1)/2}$ be $n \times n$ matrices such that these matrices are the standard orthonormal basis of symmetric $n \times n$ matrices. Then G_1 is a Wigner matrix and the hypothesis of Theorem 1.1 is satisfied with $c_2 = 2$ (see [DS01, Theorem II.11] or [Ver12, Theorem 5.32]).

Example 1.7. Let G_1, \dots, G_m be i.i.d. $n \times n$ Haar-distributed random unitary matrices. Then the random matrices G_1, \dots, G_m satisfy the hypothesis of Theorem 1.1 with $c_2 = 1$.

We give a brief overview of the strategy of the proof of Theorem 1.4 and its generalization, Theorem 3.9. In order to prove the majorization principle, we first establish some basic facts about Fourier analysis of matrix-valued functions f in Section 2. In particular, starting from a function $f: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$, we extend f to a noncommutative multilinear polynomial $Q = Q_f$ of m variables.

For these polynomials Q , we consider a few different inner products, norms, derivatives, Plancherel identities, and we also define the Ornstein-Uhlenbeck semigroup. We then prove a noncommutative hypercontractive inequality for such polynomials Q . This hypercontractive inequality is developed in Section 3 and proven in Theorem 3.1. It can also be considered a polynomial generalization of the matrix Khintchine inequality.

To prove the noncommutative majorization principle, we use the Lindeberg replacement method from [MOO10, Cha06], but in our (matrix-valued) polynomials, we are replacing one random variable with one random matrix, one at a time. The “second order” terms in this replacement have a noncommutative nature which introduces an error. Instead of trying to bound this error (which seems difficult or impossible in general), we choose particular random matrices such that the “second order” terms are small. This is accomplished by replacing an $n \times n$ random matrix X with a much larger $p \times p$ matrix $\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} H$, where H is a uniformly random $p \times p$ unitary matrix and $p > n$. When $p \rightarrow \infty$, the noncommutative “second order” errors in the Lindeberg replacement vanish.

1.2. Application: computational hardness for the noncommutative Grothendieck inequality. The *commutative* Grothendieck inequality relates two norms on a tensor product space. It can be stated as follows.

Theorem 1.8 (Grothendieck’s Inequality, [Gro53, LP68, AN06, BMMN13]). *There exists $1 \leq K_G < \frac{\pi}{2 \log(1+\sqrt{2})}$ such that the following holds. Let $n \in \mathbb{N}$ and let $A = (a_{ij})_{1 \leq i, j \leq n}$ be an $n \times n$ real matrix. Then*

$$\max_{\substack{x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}^{2n-1} \\ \langle x_i, x_i \rangle = \langle y_i, y_i \rangle = 1 \forall i \in \{1, \dots, n\}}} \sum_{i, j=1}^n a_{ij} \langle x_i, y_j \rangle \leq K_G \max_{\varepsilon_1, \dots, \varepsilon_n, \delta_1, \dots, \delta_n \in \{\pm 1\}} \sum_{i, j=1}^n a_{ij} \varepsilon_i \delta_j. \quad (3)$$

Moreover, a similar bound (with a different constant) holds for complex scalars, replacing \mathbb{R}^{2n-1} with \mathbb{C}^{2n-1} and $\{-1, 1\}$ with complex numbers of modulus 1.

For two recent surveys on Grothendieck inequalities, see [KN12] and [Pis12].

Let n, m, N be positive integers. For $x = (x_1, \dots, x_N), y = (y_1, \dots, y_N) \in \mathbb{C}^N$, define $\langle x, y \rangle := \sum_{i=1}^N x_i \bar{y}_i$. Given $U \in M_n(\mathbb{C}^N)$, define two matrices UU^* and $U^*U \in M_n(\mathbb{C})$ by

$$\forall i, j \in \{1, \dots, n\}, \quad (UU^*)_{ij} := \sum_{k=1}^N \langle U_{ik}, U_{jk} \rangle \quad (U^*U)_{ij} := \sum_{k=1}^N \langle U_{ki}, U_{kj} \rangle.$$

Define

$$\begin{aligned} \mathcal{O}_n(\mathbb{R}^N) &:= \{U \in M_n(\mathbb{R}^N) : UU^* = U^*U = I\}, \\ \mathcal{U}_n(\mathbb{C}^N) &:= \{U \in M_n(\mathbb{C}^N) : UU^* = U^*U = I\}, \end{aligned}$$

and $\mathcal{O}_n := \mathcal{O}_n(\mathbb{R})$, $\mathcal{U}_n := \mathcal{U}_n(\mathbb{C})$.

The noncommutative Grothendieck inequality (NCGI) was conjectured in [Gro53] and proven in [Pis78, Kai83]. It states the following.

Theorem 1.9 (Noncommutative Grothendieck Inequality, [Pis78, Haa85, NRV14]). *There exist constants $1 \leq K_{\mathbb{R}} \leq 2\sqrt{2}$ and $K_{\mathbb{C}} = 2$ such that the following holds. Let $n \in \mathbb{N}$ and let $M = (M_{ijkl})_{1 \leq i, j, k, \ell \leq n} \in M_n(M_n(\mathbb{C}))$. Then*

$$\sup_{U, V \in \mathcal{U}_n(\mathbb{C}^N)} \left| \sum_{i, j, k, \ell=1}^n M_{ijkl} \langle U_{ij}, V_{k\ell} \rangle \right| \leq K_{\mathbb{C}} \cdot \sup_{X, Y \in \mathcal{U}_n} \left| \sum_{i, j, k, \ell=1}^n M_{ijkl} X_{ij} \bar{Y}_{k\ell} \right|. \quad (4)$$

Let $n \in \mathbb{N}$ and let $M_{ijkl} \in M_n(M_n(\mathbb{R}))$. Then

$$\sup_{U, V \in \mathcal{O}_n(\mathbb{R}^N)} \sum_{i, j, k, \ell=1}^n M_{ijkl} \langle U_{ij}, V_{k\ell} \rangle \leq K_{\mathbb{R}} \cdot \sup_{X, Y \in \mathcal{O}_n} \sum_{i, j, k, \ell=1}^n M_{ijkl} X_{ij} Y_{k\ell}. \quad (5)$$

The constant $K_{\mathbb{C}} = 2$ is the smallest possible in (4). The smallest possible constant in (5) is still unknown [NRV14].

Remark 1.10. In Theorem 1.9, if we choose M “diagonal”, i.e. so that $M_{ijkl} = M_{iikk}$ for all $i, j, k, \ell \in \{1, \dots, n\}$, then we recover Theorem 1.8. That is, (5) is a generalization of (3).

Remark 1.11. For $U, V \in M_n(\mathbb{C}^N)$ vector-valued matrices define $U \odot V \in M_{n \times n}(\mathbb{C})$ by $(U \odot V)_{(i-1)n+k, (j-1)n+\ell} := \langle U_{ij}, V_{k\ell} \rangle$ for all $i, j, k, \ell \in \{1, \dots, n\}$. Let $M \in M_{n \times n}(\mathbb{C})$. Then (4) has the equivalent form

$$\sup_{U, V \in \mathcal{U}_n(\mathbb{C}^N)} |\mathrm{Tr}(M(U \odot V))| \leq K_{\mathbb{C}} \cdot \sup_{X, Y \in \mathcal{U}_n} |\mathrm{Tr}(M(X \otimes \bar{Y}))|,$$

which is the formulation we will mainly use in the paper.

The commutative Grothendieck inequality, Theorem 1.8, was given an algorithmic interpretation in [AN06], where it is shown that the left-hand side of (3) can be computed in polynomial time and any optimal solution “rounded” to a near-optimal (up to the approximation constant) choice of signs $\varepsilon_1, \dots, \varepsilon_n, \delta_1, \dots, \delta_n$ for the right-hand side; see also [BMMN13, NR14] for more recent works. Moreover, assuming a standard conjecture in complexity theory, the *Unique Games Conjecture* (UGC, see Section 2.12 for the definition), Raghavendra and Steurer [RS09] showed that no better approximation to the left-hand side of (3) could be computed in polynomial time, unless $P=NP$.¹

¹ Other sharp Unique Games hardness results for variants of (3) were proven by Khot and Naor [KN09, KN13].

Following the same path, in [NRV14] it was shown that the noncommutative Grothendieck inequality, Theorem 1.9, could be made algorithmic as well, in the sense that the left-hand side of each inequality can be efficiently approximated in polynomial time. And near-optimal solutions can be efficiently “rounded” to an assignment of variables to the right-hand side with a multiplicative loss in the objective value corresponding to the constants $K_{\mathbb{C}}$ and $K_{\mathbb{R}}$ respectively. Furthermore, very recently Briët, Regev and Saket [BRS15] showed that approximations within a factor less than $K_{\mathbb{C}} = 2$ in (4) cannot be found in polynomial time unless $P=NP$. Their proof replace the standard “dictatorship versus low influences” machinery with a clever construction of a linear transformation using the Clifford algebra, thereby entirely avoiding the use of invariance principles, majorization principles and hypercontractivity. In fact, as [BRS15] mentions, “Our attempts to apply these techniques here failed.”

Our main application for our moment majorization theorem states that the approximation given in (4) is the best achievable in polynomial time, assuming the Unique Games conjecture. Although this is a weaker result than [BRS15] (since they do not need to assume UGC), our proof arguably has the advantage of following the same general structure as the hardness result for the commutative Grothendieck inequality proved in [RS09]. As such our proof technique demonstrates that the noncommutative generalizations of the majorization and hypercontractivity principles provided in this paper can lead to successful extensions of their use as key tools in hardness of approximation results and more generally analysis of Boolean functions. We expect the principles to find more applications; for example, it is relatively easy for us to prove Theorem 1.16 below by imitating the proof of Theorem 1.14. The proof of [BRS15] seems to provide no such flexibility. (In light of our results it would nevertheless be interesting to try to connect their methods with the hypercontractive inequality of [CL93], since both use Clifford algebras.)

Theorem 1.12 (Unique Games Hardness for NCGI). *Assuming the Unique Games Conjecture, it is NP-hard to approximate the quantity*

$$\text{OPT}(M) := \sup_{X, Y \in \mathcal{U}_n} \left| \sum_{i, j, k, \ell=1}^n M_{ijkl} X_{ij} \overline{Y_{k\ell}} \right| \quad (6)$$

within a multiplicative factor smaller than $K_{\mathbb{C}} = 2$.

The proof of Theorem 1.12 considers a restricted variant where M is a positive semidefinite 4-tensor, in the following sense.

Definition 1.13. Let $M \in M_{n \times n}(\mathbb{C}) = M_n(M_n(\mathbb{C}))$. We say that M is positive semidefinite, or PSD (as a 4-tensor), if there exists $M_1, \dots, M_N \in M_n(\mathbb{C})$ such that $M = \sum_{i=1}^N M_i \otimes \overline{M_i}$.

Theorem 1.14 (Unique Games Hardness for Positive Semidefinite NCGI). *Let $K > 0$ be the infimum over all constant $K_{\mathbb{C}}$ such that (4) holds for all $M \in M_{n \times n}(\mathbb{C})$ such that M is PSD. Then, assuming the Unique Games Conjecture, no polynomial time algorithm (in n) can approximate the quantity*

$$\text{OPT}(M) = \sup_{X, Y \in \mathcal{U}_n} \left| \sum_{i, j, k, \ell=1}^n M_{ijkl} X_{ij} \overline{Y_{k\ell}} \right|$$

within a multiplicative factor smaller than K .

As shown in [BRS15, Theorem 1.2] the constant K in Theorem 1.14 satisfies $K = K_C = 2$, so that Theorem 1.12 follows from Theorem 1.14.

We consider a last variant of NCGI, introduced in [BKS16].

Theorem 1.15 (Positive Semidefinite Variant of NCGI). *Let $d \in \mathbb{N}$. Then there exists $K(d) > 0$ such that the following holds. Let $n \in \mathbb{N}$ and let M be a symmetric positive semidefinite $nd \times nd$ complex matrix. (That is, M is positive semidefinite in the usual sense.) For each $i, j \in \{1, \dots, n\}$, let (M_{ij}) denote the $d \times d$ matrix $\{M_{d(i-1)+u, d(j-1)+v}\}_{u,v=1}^d$. Then*

$$\sup_{\substack{V_1, \dots, V_n \in \mathbb{C}^{d \times dn} \\ V_i V_i^* = 1_{d \times d} \\ \forall i \in \{1, \dots, n\}}} \sum_{i,j=1}^n \text{Tr}((M_{ij})^T V_i V_j^*) \leq K(d) \cdot \sup_{X_1, \dots, X_n \in \mathcal{U}_d} \sum_{i,j=1}^n \text{Tr}((M_{ij})^T X_i X_j^*).$$

As a demonstration of their flexibility, the proofs of Theorem 1.12 and Theorem 1.14 readily extends to this context.

Theorem 1.16 (Unique Games Hardness for Positive Semidefinite NCGI Variant). *Let M be as in Theorem 1.15. Let K be the infimum over all $K(d) > 0$ such that Theorem 1.15 holds. Then, assuming the Unique Games Conjecture, no polynomial time (in n) algorithm can approximate the quantity*

$$\sup_{X_1, \dots, X_n \in \mathcal{U}_d} \sum_{i,j=1}^n \text{Tr}((M_{ij})^T X_i X_j^*).$$

within a multiplicative factor smaller than K .

As shown in [BKS16], we have $\sqrt{K(d)} = \mathbb{E} \left(\frac{1}{d} \sum_{i=1}^d \lambda_i(G) \right)$, where G is a $d \times d$ matrix with complex Gaussian i.i.d. entries with mean zero and variance $1/d$, and $\lambda_i(G)$ is the i^{th} singular value of G . It is known that $K(1) = \pi/4$ and $\lim_{d \rightarrow \infty} K(d) = (8/(3\pi))^2$.

The proof of Theorems 1.14 and 1.16 rely on our noncommutative majorization principle. They are given in Section 5.

It seems conceivable that Theorems 1.14 and 1.16 can be extended to handle real scalars instead of complex scalars. We leave this research direction to future investigation.

1.3. Other applications. The commutative invariance principle [Rot79, Cha06, MOO10] implies that if Q is a commutative multilinear polynomial with small derivatives (i.e. small influences), then the distribution of Q on i.i.d. uniform inputs in $\{-1, 1\}$ is close to the distribution of Q on i.i.d. standard Gaussian random variables. A more general statement can be made for more general functions and distributions; for details see e.g. [MOO10]. An invariance principle can also be considered as a concentration inequality, generalizing the central limit theorem with error bounds (i.e. the Berry-Esséen Theorem). For other variants of invariance principles, see [Mos10, IM12].

The form of the invariance principle given in [MOO10] is proven by a combination of the Lindeberg replacement argument and the hypercontractive inequality [Bon70, Nel73, Gro75]. That is, one replaces one argument of Q at a time, adding up the resulting errors and controlling them via the hypercontractive inequality. One version of hypercontractivity says that a higher L_q norm of a polynomial is bounded by a lower L_p norm of that polynomial,

where $q > p$, with a bound dependent on the degree of the polynomial Q . For example, if Q has degree d , then the L_4 norm of Q is bounded by 9^d times the L_2 norm of Q .

The commutative invariance principle has seen many applications [O’D14b, O’D14a] in recent years. Here is a small sample of such applications and references: isoperimetric problems in Gaussian space and in the hypercube [MOO10, IM12], social choice theory, Unique Games hardness results [KKMO07, IM12], analysis of algorithms [BR15], random matrix theory [MP14], free probability [NPR10], optimization of noise sensitivity [Kan14]. The Lindeberg replacement argument itself has many applications, e.g. in proving the universality of eigenvalue statistics for Wigner matrices [TV11, Theorem 15].

We anticipate that our noncommutative majorization principle will find similar applications. Even though it is impossible to prove a noncommutative invariance principle in general, most applications of the commutative invariance principle only involve one direction of the inequality. That is, most applications of the invariance principle are really just applications of a majorization principle such as Theorem 1.1 or 1.4.

To demonstrate further applications of our majorization principle, we show in Section 6 that one of the two main parts of the proof of the Majority is Stablest Theorem from [MOO10] can be extended to the noncommutative setting. Then, in Section 7, we demonstrate a (probably sub-optimal) anti-concentration estimate for noncommutative multilinear polynomials. Both of these results proceed as in [MOO10] by replacing their invariance principle with our majorization principle.

Since majorization principles such as Theorems 1.1 and 1.4 show the closeness of one distribution to another, these statements could be fit into the “concentration of measure” paradigm. The paper [MS12] proves a concentration inequality for noncommutative polynomials, but these methods seem insufficient to prove a majorization principle. An invariance principle has been proven in the free probability setting [DN14], but the details of exactly what polynomials can be dealt with, and which distributions can be handled, seem incomparable to our majorization principle.

Remark 1.17. We remark that, although it may be tempting to try to prove Theorem 1.1 from the commutative invariance principle of [MOO10], there seems to be no straightforward way to accomplish this task. For example, we could interpret each entry in the output of a noncommutative polynomial Q as a commutative polynomial function of the inputs. But then in order to control $\text{Tr} |Q|^4$, we would need information on the joint distribution of the entries of Q , which is not provided by the invariance principle of [MOO10].

2. DEFINITIONS, BACKGROUND AND NOTATION

2.1. Matrices. For $n \in \mathbb{N}$ we denote the set of n by n matrices by $M_n(\mathbb{C})$. We use $M_{n \times n}(\mathbb{C})$ in place of M_{n^2} to denote n^2 by n^2 matrices when we wish to emphasize that a specific tensor decomposition of the space $\mathbb{C}^{n^2} = \mathbb{C}^n \otimes \mathbb{C}^n$ on which the matrix acts has been fixed.

For $A \in M_n(\mathbb{C})$, $\|A\|$ is the operator norm of A (the largest singular value). We denote by A^* the conjugate-transpose. The absolute value is $|A| = (AA^*)^{1/2}$. We use I to denote the identity matrix.

Any real function $f : \mathbb{R} \rightarrow \mathbb{R}$ can be applied to a Hermitian matrix A by diagonalizing A and applying f to the eigenvalues of A . Define $\text{Chop} : \mathbb{R} \rightarrow \mathbb{R}$ as $\text{Chop}(t) = t$ if $|t| \leq 1$, $\text{Chop}(t) = 1$ if $t \geq 1$, and $\text{Chop}(t) = -1$ if $t \leq -1$.

2.2. Asymptotic Notation. Let $a, b, c \in \mathbb{R}$. We write $a = O_c(b)$ if there exists a constant $C(c) > 0$ such that $|a| \leq C(c) |b|$. We write $a \leq O_c(b)$ if there exists a constant $C(c) > 0$ such that $a \leq C(c)b$.

2.3. Random variables and expectations. The following notational conventions will be useful when working with functions of multiple variables. Let $m, n \in \mathbb{N}$ and let S, T be arbitrary sets. Let $f : S^m \rightarrow T$ and let X_1, \dots, X_n be independent random variables with the same distribution \mathcal{X} taking values in S . Then we will denote $\mathbb{E}f(X_1, \dots, X_m)$ by $\mathbb{E}_{X_j \sim \mathcal{X}} f\{X_j\}$; more generally the curly bracket notation $f\{A_j\}$ will be used to denote $f(A_1, \dots, A_m)$.

We will use the following ensembles. Let $n, N \in \mathbb{N}$, and let p be a positive multiple of n . $H \sim \mathcal{H}$ denotes a $p \times p$ Haar-distributed random unitary matrix, where the dimension p will always be clear from context. $b \sim \mathcal{B}$ denotes a uniformly random $b \in \{\pm 1\}$. $G \sim \mathcal{G}$ denotes any random variable taking values in $M_n(\mathbb{C})$ such that $\mathbb{E}G = 0$ and $\mathbb{E}GG^* = I$. And $G' \sim \mathcal{G} \otimes I$ denotes $G' = G \otimes I_{p/n} \in M_p(\mathbb{C})$, where $G \sim \mathcal{G}$. $G \sim \mathcal{V}$ denotes a random variable distributed as $G = \sum_{i=1}^N g_i V_i$, where g_1, \dots, g_N are i.i.d. standard complex Gaussian random variables and V_1, \dots, V_N are $n \times n$ complex matrices satisfying $\sum_{i=1}^N V_i V_i^* = \sum_{i=1}^N V_i^* V_i = I_n$. And $G' \sim \mathcal{V} \otimes I$ denotes $G' = G \otimes I_{p/n} \in M_p(\mathbb{C})$, where $G \sim \mathcal{V}$. Whenever \mathcal{V} is used the matrices V_1, \dots, V_N will be clear from context. We also sometimes write $G_j \sim \mathcal{D}$ to mean G_1, \dots, G_m are independent random variables with distribution \mathcal{D} , where again m will always be clear from context.

2.4. Fourier expansions. Let $n, m \in \mathbb{N}$ and $f, h : \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$. We consider the inner product

$$\langle f, h \rangle := \mathbb{E}_{b_j \sim \mathcal{B}} \text{Tr}(f\{b_j\}(h\{b_j\})^*) = 2^{-m} \sum_{\sigma \in \{-1, 1\}^m} \text{Tr}(f(\sigma)h(\sigma)^*).$$

Given $S \subseteq \{1, \dots, m\}$ and $\sigma = (\sigma_1, \dots, \sigma_m) \in \{-1, 1\}^m$, define

$$W_S(\sigma) := \prod_{i \in S} \sigma_i.$$

The set of functions $\{W_S\}_{S \subseteq \{1, \dots, m\}}$ forms an orthonormal basis for the space of functions from $\{-1, 1\}^m$ to $M_n(\mathbb{C})$, when it is viewed as a vector space over \mathbb{C} with respect to the inner product $\langle \cdot, \cdot \rangle$.

Let $\hat{f}(S) := 2^{-m} \sum_{\sigma \in \{-1, 1\}^m} f(\sigma)W_S(\sigma)$ be the Fourier coefficient of f associated to S . Note that $\hat{f}(S) \in M_n(\mathbb{C})$. Then $f = \sum_{S \subseteq \{1, \dots, m\}} \hat{f}(S)W_S$, and

$$\begin{aligned} 2^{-m} \sum_{\sigma \in \{-1, 1\}^m} f(\sigma)(g(\sigma))^* &= 2^{-m} \sum_{\sigma \in \{-1, 1\}^m} \sum_{S, S' \subseteq \{1, \dots, m\}} \hat{f}(S)(\hat{g}(S'))^* W_S(\sigma)W_{S'}(\sigma) \\ &= \sum_{S, S' \subseteq \{1, \dots, m\}} \hat{f}(S)(\hat{g}(S'))^* \cdot 2^{-m} \sum_{\sigma \in \{-1, 1\}^m} W_S(\sigma)W_{S'}(\sigma) \\ &= \sum_{S \subseteq \{1, \dots, m\}} \hat{f}(S)(\hat{g}(S))^*. \end{aligned} \tag{7}$$

2.5. Noncommutative polynomials. Let $n, m \in \mathbb{N}$ be integers. We consider noncommutative multilinear polynomials $Q \in M_n(\mathbb{C})[X_1, \dots, X_m]$, where X_1, \dots, X_m are noncommutative indeterminates. Monomials are always ordered by increasing order of the index, e.g. X_1X_2 and not X_2X_1 . Any such polynomial can be expanded as

$$Q(X_1, \dots, X_m) = \sum_{S \subseteq \{1, \dots, m\}: |S| \leq d} \widehat{Q}(S) \prod_{i \in S} X_i, \quad (8)$$

where $\widehat{Q}(S) \in M_n(\mathbb{C})$ for all $S \subseteq \{1, \dots, m\}$ and $0 \leq d \leq m$ is the degree of Q , defined as $\max\{|S| : \widehat{Q}(S) \neq 0, S \subseteq \{1, \dots, m\}\}$.

Let $f: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$. Define the (non-commutative) *multilinear polynomial* $Q_f \in M_n(\mathbb{C})[X_1, \dots, X_m]$ associated to f by

$$Q_f(X_1, \dots, X_m) := \sum_{S \subseteq \{1, \dots, m\}} \widehat{f}(S) \prod_{i \in S} X_i. \quad (9)$$

2.6. Partial Derivatives. Let $f: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$. Let $i \in \{1, \dots, m\}$. Define the i^{th} *partial derivative* of f by

$$\partial_i f(\sigma) := \sum_{S \subseteq \{1, \dots, m\}: i \in S} \widehat{f}(S) W_S(\sigma) = \frac{1}{2}(f(\sigma) - f(\sigma_1, \dots, -\sigma_i, \dots, \sigma_m)), \quad (10)$$

and the i^{th} *influence* of f by

$$\text{Inf}_i f := \sum_{S \subseteq \{1, \dots, m\}: i \in S} \text{Tr}(\widehat{f}(S) \widehat{f}(S)^*). \quad (11)$$

Note that by (7), $\text{Inf}_i f = \langle \partial_i f, \partial_i f \rangle$ and $\sum_{i=1}^m \text{Inf}_i f = \sum_{S \subseteq \{1, \dots, m\}} |S| \text{Tr}(\widehat{f}(S) (\widehat{f}(S))^*)$.

2.7. Ornstein-Uhlenbeck semigroup. For $f, h: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$ define their convolution $f * h$ by

$$f * h(\sigma) := 2^{-m} \sum_{\omega \in \{-1, 1\}^m} f(\sigma \cdot \omega^{-1}) h(\omega) = \sum_{S \subseteq \{1, \dots, m\}} \widehat{f}(S) \widehat{h}(S) W_S(\sigma), \quad \forall \sigma \in \{-1, 1\}^m. \quad (12)$$

Here $\sigma \cdot \omega$ denotes the componentwise product of σ and ω and ω^{-1} denotes the multiplicative inverse of ω , so that $\omega \cdot \omega^{-1} = (1, \dots, 1)$.

Let $0 < \rho < 1$. For any $\sigma = (\sigma_1, \dots, \sigma_m) \in \{-1, 1\}^m$, let

$$R_\rho(\sigma) := \prod_{j=1}^m (1 + \rho \sigma_j) = \sum_{S \subseteq \{1, \dots, m\}} \rho^{|S|} W_S(\sigma). \quad (13)$$

Let $f: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$. Define the Ornstein-Uhlenbeck semigroup $T_\rho f$ by

$$T_\rho f(\sigma) := \sum_{S \subseteq \{1, \dots, m\}} \rho^{|S|} \widehat{f}(S) W_S(\sigma) = f * R_\rho(\sigma), \quad \forall \sigma \in \{-1, 1\}^m. \quad (14)$$

2.8. Truncation of Fourier Coefficients (or Littlewood-Paley Projections). Let $f: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$, and $Q_f \in M_n(\mathbb{C})[X_1, \dots, X_m]$ the multilinear polynomial associated to f . Let $d \in \mathbb{N}$. Let P_d denote *projection onto the level- d Fourier coefficients*. That is, $\forall \sigma \in \{-1, 1\}^m, \forall X_1, \dots, X_m \in M_n(\mathbb{C})$,

$$P_d f(\sigma) := \sum_{S \subseteq \{1, \dots, m\}: |S|=d} \widehat{f}(S) W_S(\sigma), \quad P_d Q_f(X_1, \dots, X_m) := \sum_{S \subseteq \{1, \dots, m\}: |S|=d} \widehat{f}(S) \prod_{i \in S} X_i.$$

Let $P_{\leq d} := \sum_{i \leq d} P_i$ denote projection onto the Fourier coefficients of degree at most d . Denote $P_{> d} f := f - P_{\leq d} f$, $P_{> d} Q_f := Q_f - P_{\leq d} Q_f$.

2.9. Embeddings.

Definition 2.1 (Matrix embedding). Let $A \in M_n(\mathbb{C})$ and $p \geq n$. Define the embedding $\iota_p: M_n(\mathbb{C}) \rightarrow M_p(\mathbb{C})$ by

$$\iota_p(A) := \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in M_p(\mathbb{C}).$$

If $B = \sum_{k=1}^N C_k \otimes D_k \in M_{n \times n}(\mathbb{C})$, extend this definition to

$$\iota_p(B) := \sum_{k=1}^N \iota_p(C_k) \otimes \iota_p(D_k) \in M_{p \times p}(\mathbb{C}).$$

Lastly, if $f: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$ define $\iota_p(f): \{-1, 1\}^m \rightarrow M_p(\mathbb{C})$ by

$$\iota_p(f)(\sigma) = \iota_p(f(\sigma)) \quad \forall \sigma \in \{-1, 1\}^m.$$

We will sometimes denote the same quantities by A^ι , B^ι and f^ι respectively, leaving the dependence of ι on p and n implicit for clarity of notation.

Note that if $Q = \sum_{S \subseteq \{1, \dots, m\}} \widehat{Q}(S) \prod_{i \in S} X_i \in M_n(\mathbb{C})[X_1, \dots, X_m]$ is a noncommutative polynomial the last item in Definition 2.1 is equivalent to defining

$$Q^\iota = \sum_{S \subseteq \{1, \dots, m\}} \iota_p(\widehat{Q}(S)) \prod_{i \in S} X_i \in M_p(\mathbb{C})[X_1, \dots, X_m].$$

If moreover Q_f is defined from $f: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$ as in (9) then

$$Q_f^\iota = \sum_{S \subseteq \{1, \dots, m\}} \iota_p(\widehat{f}(S)) \prod_{i \in S} X_i \in M_p(\mathbb{C})[X_1, \dots, X_m].$$

2.10. Coordinate projections.

Definition 2.2. Let $U \in M_n(\mathbb{C}^N)$ with $UU^* = U^*U = I$ and $Q \in M_n(\mathbb{C})[X_1, \dots, X_m]$ a noncommutative polynomial. We denote the *Gaussian L_2 norm* of Q associated to the ensemble \mathcal{G} by

$$\|Q\|_{L_2, \mathcal{G}} := \left(\mathbb{E}_{\substack{G_j \sim \mathcal{G} \\ H_j \sim \mathcal{H}_p}} \text{Tr} |Q^\iota \{G_j H_j\}|^2 \right)^{1/2}.$$

Let $\mathcal{N}(0, 1)$ denote the standard complex Gaussian distribution for a random variable. For integers m, N let $\mathbb{E}_{g_{i,j} \sim \mathcal{N}(0,1)}$ denote expectation with respect to $g_{1,1}, \dots, g_{m,N}$ i.i.d. $\mathcal{N}(0, 1)$. For any function R in the mN random variables $g_{i,j} \sim \mathcal{N}(0, 1)$, for any $i = 1, \dots, m$, let

$$\mathcal{P}_i R(g_{1,1}, \dots, g_{m,N}) = \sum_{j=1}^N \mathbb{E}_{g'_{i,j} \sim \mathcal{N}(0,1)} [(R\{g'_{i,j}\}) \overline{g'_{i,j}}] g_{i,j} \quad (15)$$

be the projection of R onto the linear span of the $\{g_{i,j}\}_{j=1, \dots, N}$. The projection \mathcal{P}_i is naturally extended to noncommutative polynomials $R \in M_n(\mathbb{C})[X_1, \dots, X_m]$ by applying it to each matrix entry of R (when the variables X_1, \dots, X_m are themselves matrix-valued functions of the $g_{i,j}$). We note the following facts:

Lemma 2.3. *Let $b: (M_p(\mathbb{C}))^m \rightarrow M_p(\mathbb{C})$ satisfy $\|b(x)\| \leq 1$ for all $x \in (M_p(\mathbb{C}))^m$.*

(1) *For any PSD matrix $M \in M_{n \times n}(\mathbb{C})$,*

$$\begin{aligned} & \mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H_j, J_j \sim \mathcal{H}_p}} \sum_{i=1}^m \text{Tr}(M^\iota \cdot (\mathcal{P}_i b\{G_j H_j\} \otimes \overline{\mathcal{P}_i b\{G_j J_j\}}) \cdot (H_i^* \otimes \overline{J_i^*})) \\ & \leq \mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H, J \sim \mathcal{H}_p}} \text{Tr}(M^\iota \cdot (b\{G_j H\} \otimes \overline{b\{G_j J\}}) \cdot (H^* \otimes \overline{J^*})). \end{aligned}$$

(2) $\|\sum_{i=1}^m \mathcal{P}_i b\|_{L_2, \mathcal{G}} \leq \|b\|_{L_2, \mathcal{G}}$.

Proof. We begin with (1). Recall that \mathcal{V} is defined so that $G_j = \sum_{i=1}^N g_{i,j} V_i = \langle g_j, V \rangle$, where $\{g_{i,j}\}_{1 \leq i \leq N, 1 \leq j \leq m}$ are i.i.d. standard complex Gaussian random variables and $g_j = (g_{j,1}, \dots, g_{j,N})$ for any $j \in \{1, \dots, m\}$. By a density argument, it suffices to prove (1) when b is a polynomial in the entries of its matrix variables. Then we can write

$$b(\langle g_1, V \otimes I \rangle H, \dots, \langle g_m, V \otimes I \rangle H) = \sum_{a=(a_1, \dots, a_m) \in \{1, 2, \dots, N\}^m} r_a \prod_{i=1}^m g_{i, a_i}^{d_i}, \quad r_a \in M_p(\mathbb{C}), d_i \in \mathbb{N}. \quad (16)$$

where the coefficients r_a do not depend on g_1, \dots, g_m , but they can depend on H . Note that for any $i \in \{1, \dots, m\}$ and for any $j \in \{1, \dots, N\}$, if $e_{ij} \in \mathbb{N}^m$ denotes the vector whose only nonzero entry is a j in the i^{th} entry, then

$$\mathbb{E}_{g_1, \dots, g_m} b(\langle g_1, V \otimes I \rangle H, \dots, \langle g_m, V \otimes I \rangle H) g_{i,j}^* = r_{e_{ij}}.$$

So,

$$\sum_{j=1}^N [\mathbb{E}_{g'_1, \dots, g'_m} b(\langle g'_1, V \otimes I \rangle H, \dots, \langle g'_m, V \otimes I \rangle H) (g'_{i,j})^*] g_{i,j} = \sum_{j=1}^N r_{e_{ij}} g_{i,j}.$$

We can express $b(\langle g_1, V \otimes I \rangle H, \dots, \langle g_m, V \otimes I \rangle H) H^*$ as the sum of two terms A and B such that A contains only linear terms in $g_{i,j}$ where $1 \leq i \leq m$ and $1 \leq j \leq N$, and B contains

only higher order terms in $g_{i,j}$, as follows,

$$\begin{aligned}
& b(\langle g_1, V \otimes I \rangle H, \dots, \langle g_m, V \otimes I \rangle H) H^* \\
&= \sum_{i=1}^m \sum_{j=1}^N \left[\mathbb{E}_{g'_1, \dots, g'_m} b(\langle g'_1, V \otimes I \rangle H, \dots, \langle g'_m, V \otimes I \rangle H) (g'_{i,j})^* \right] g_{i,j} H^* \\
&+ \left(b(\langle g_1, V \otimes I \rangle H, \dots, \langle g_m, V \otimes I \rangle H) H^* \right. \\
&\quad \left. - \sum_{i=1}^m \sum_{j=1}^N \left[\mathbb{E}_{g'_1, \dots, g'_m} b(\langle g'_1, V \otimes I \rangle H, \dots, \langle g'_m, V \otimes I \rangle H) (g'_{i,j})^* \right] g_{i,j} H^* \right),
\end{aligned}$$

where A corresponds to the term on the second line and B to the terms on the third and fourth line. For g a standard complex Gaussian random variable, we have $\mathbb{E}|g|^2 g^k = 0$ for any positive integer k , thus $\mathbb{E}(A \otimes \bar{B}) = \mathbb{E}(B \otimes \bar{A}) = 0$, and

$$\mathbb{E}\text{Tr}(\iota(M)(A + B)) \otimes \overline{(A + B)} = \mathbb{E}\text{Tr}(\iota(M)(A \otimes \bar{A})) + \mathbb{E}\text{Tr}(\iota(M)(B \otimes \bar{B})).$$

Since M is PSD, $\iota(M)$ is PSD, so we have $\mathbb{E}\text{Tr}(\iota(M)(B \otimes \bar{B})) \geq 0$, so that

$$\begin{aligned}
& \mathbb{E}\text{Tr}(\iota(M) \cdot (b(\langle g_1, V \otimes I \rangle H, \dots, \langle g_m, V \otimes I \rangle H) H^* \otimes \overline{b(\langle g_1, V \otimes I \rangle J, \dots, \langle g_m, V \otimes I \rangle J) J^*})) \\
&\geq \sum_{i=1}^m \mathbb{E}\text{Tr}(\iota(M) \cdot \left(\sum_{j=1}^N \left[\mathbb{E}_{g'_1, \dots, g'_m} b(\langle g'_1, V \otimes I \rangle H, \dots, \langle g'_m, V \otimes I \rangle H) (g'_{i,j})^* \right] g_{i,j} H^* \right. \\
&\quad \left. \otimes \overline{\sum_{j=1}^N \left[\mathbb{E}_{g'_1, \dots, g'_m} b(\langle g'_1, V \otimes I \rangle J, \dots, \langle g'_m, V \otimes I \rangle J) (g'_{i,j})^* \right] g_{i,j} J^*} \right)) \\
&= \sum_{i=1}^m \mathbb{E}\text{Tr}(\iota(M) \cdot (\mathcal{P}_i b(\langle g'_1, V \otimes I \rangle H_1, \dots, \langle g'_m, V \otimes I \rangle H_m) H_i^* \\
&\quad \otimes \overline{\mathcal{P}_i b(\langle g'_1, V \otimes I \rangle J_1, \dots, \langle g'_m, V \otimes I \rangle J_m) J_i^*})),
\end{aligned}$$

where the last equality uses the definition (15) of \mathcal{P}_i and the fact that $\mathcal{P}_i b$ only depends on the i^{th} variable of b , so that

$$\mathbb{E}_{H_i \sim \mathcal{H}} \mathcal{P}_i b\{\langle g'_j, V \otimes I \rangle H_j\} H_i^* = \mathbb{E}_{H \sim \mathcal{H}} \mathcal{P}_i b\{\langle g'_j, V \otimes I \rangle H\} H^*.$$

Item (2) is proven similarly, expanding

$$\begin{aligned}
& b(\langle g_1, V \otimes I \rangle H_1, \dots, \langle g_m, V \otimes I \rangle H_m) \\
&= \sum_{i=1}^m \sum_{j=1}^N \left[\mathbb{E}_{g'_1, \dots, g'_m} b(\langle g'_1, V \otimes I \rangle H_1, \dots, \langle g'_m, V \otimes I \rangle H_m) (g'_{i,j})^* \right] g_{i,j} \\
&+ \left(b(\langle g_1, V \otimes I \rangle H_1, \dots, \langle g_m, V \otimes I \rangle H_m) \right. \\
&\quad \left. - \sum_{i=1}^m \sum_{j=1}^N \left[\mathbb{E}_{g'_1, \dots, g'_m} b(\langle g'_1, V \otimes I \rangle H_1, \dots, \langle g'_m, V \otimes I \rangle H_m) (g'_{i,j})^* \right] g_{i,j} \right)
\end{aligned}$$

and proceeding as in the proof of (1), so that A is the term on the second line and B the terms on the third and fourth lines above, and we use

$$\mathbb{E}\text{Tr}((A+B)(A+B)^*) = \mathbb{E}\text{Tr}(AA^*) + \mathbb{E}\text{Tr}(BB^*) \geq \mathbb{E}\text{Tr}(AA^*).$$

□

2.11. Bounds on random polynomials. The key difference between the random matrices $G_i \sim \mathcal{G}$ and $(G_i \otimes I)H_i$ where $H_i \sim \mathcal{H}_p$ is that the matrices $(G_i \otimes I)H_i$ behave well with respect to matrix products. This property is exploited in Corollary 2.5 below.

Lemma 2.4. *Let p be a positive multiple of n , let $A, B \in M_p(\mathbb{C})$. Let $I = I_n$ be the $n \times n$ identity matrix. Then*

$$\mathbb{E}_{H \sim \mathcal{H}_p} \|I^t A H B I^t\|^2 \leq \frac{n^2}{p} \|A\|^2 \|B\|^2.$$

Proof. The nonzero eigenspace K of $H B I^t B^* H^*$ is a uniformly distributed subspace of dimension at most n of \mathbb{C}^p . Given a unit vector x , the squared norm of the projection of x on K has expectation at most n/p . Applying this to the eigenvectors of $A^* I^t A$,

$$\mathbb{E}_{H \sim \mathcal{H}_p} \text{Tr}(I^t A H B I^t B^* H^* A^*) \leq \frac{n}{p} \|B\|^2 \text{Tr}(A^* I^t A) \leq \frac{n^2}{p} \|B\|^2 \|A\|^2.$$

□

Corollary 2.5. *Let $R, S \in M_n(\mathbb{C})[X_1, \dots, X_m]$ be multilinear polynomials not depending on the j -th variable such that $\mathbb{E}_{b_j \sim \mathcal{B}} \frac{1}{n} \text{Tr}|R\{b_j\}|^2 \leq 1$ and $\mathbb{E}_{b_j \sim \mathcal{B}} \frac{1}{n} \text{Tr}|S\{b_j\}|^2 \leq 1$. Then*

$$\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \left\| (S^t \{G_i H_i\}) G_j H_j (R^t \{G_i H_i\})^* \right\| = O_{n,m}(p^{-1/2}),$$

where the implicit constant may depend on n and m .

Proof. Write $S^t = S_0 + \sum_{k \neq j} S_k X_k$ and $R^t = R_0 + \sum_{k \neq j} R_k X_k$, where $\forall k \in \{1, \dots, m\} \setminus \{j\}$, $S_k, R_k \in M_n(\mathbb{C})[X_1, \dots, X_m]$ depend neither on the k -th or the j -th variable, and $S_0, R_0 \in M_n(\mathbb{C})$. Then for any $k \neq j$, since $S = I^t S$ and $R^* = R^* I^t$,

$$\begin{aligned} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \|S^t G_j H_j H_k^* G_k^* (R_k^t)^*\| &= \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \|I^t S^t G_j H_j H_k^* G_k^* (R_k^t)^* I^t\| \\ &\leq \frac{n}{\sqrt{p}} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \|S^t G_j\| \|G_k^* (R_k^t)^*\| \\ &\leq \frac{n}{\sqrt{p}} \left(\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \|S^t G_j\|^2 \right)^{1/2} \left(\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \|G_k^* (R_k^t)^*\|^2 \right)^{1/2} \\ &\leq \frac{n^2}{\sqrt{p}} \left(\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \|S^t\|^2 \right)^{1/2} \left(\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \|R_k^t\|^2 \right)^{1/2} \end{aligned}$$

where the first inequality used Lemma 2.4, and the last inequality used $\mathbb{E}_{G \sim \iota(G)} \|G\|^2 = \mathbb{E}_{G \sim \mathcal{G}} \|G\|^2 \leq \mathbb{E}_{G \sim \mathcal{G}} \text{Tr}(GG^*) = n$. To conclude, we use the normalization assumption on R, S , and (17) and (20) from Lemma 2.6 below. □

Lemma 2.6. Let $f: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$ and $Q \in M_n(\mathbb{C})[X_1, \dots, X_m]$ the multilinear polynomial associated to f . Let $0 \leq k \leq m$. Let $G_i \sim \mathcal{G}$ and $b_i \sim \mathcal{B}$. Let $\mathcal{X} = (G_1, \dots, G_k)$ and $\mathcal{Y} = (b_{k+1}, \dots, b_m)$. Then

$$\mathbb{E}_{x_i \sim \mathcal{X}, y_j \sim \mathcal{Y}} |Q\{x_i, y_j\}|^2 = \mathbb{E}_{b_i \sim \mathcal{B}} |Q\{b_i\}|^2. \quad (17)$$

$$\text{Inf}_i(Q') = \text{Inf}_i(Q), \quad \forall i \in \{1, \dots, m\}. \quad (18)$$

$$\frac{1}{n} \mathbb{E}_{b_i \sim \mathcal{B}} \text{Tr} |Q'\{b_i\}|^2 = \frac{1}{n} \mathbb{E}_{b_i \sim \mathcal{B}} \text{Tr} |Q\{b_i\}|^2. \quad (19)$$

$$\frac{1}{n} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} |Q'\{G_i H_i\}|^2 = \frac{1}{n} \mathbb{E}_{G_i \sim \mathcal{G}} \text{Tr} |Q\{G_i\}|^2. \quad (20)$$

Proof. Recall the variables $G_i \sim \mathcal{G}$ are independent, $\mathbb{E}G_i = 0$ and $\mathbb{E}G_i G_i^* = I$ for all $1 \leq i \leq k$. Similarly, the $b_i \sim \mathcal{B}$ are independent with $\mathbb{E}b_i = 0$ and $\mathbb{E}b_i b_i^* = 1$ for all $k+1 \leq i \leq m$. So,

$$\begin{aligned} & \mathbb{E}_{x \sim \mathcal{X}, y \sim \mathcal{Y}} (Q(x, y)(Q(x, y))^*) \\ &= \mathbb{E} \sum_{S \subseteq \{1, \dots, m\}} \left(\widehat{Q}(S) \left(\prod_{i \in S: i \leq k} G_i \prod_{i \in S: i > k} b_i \right) \left(\prod_{i \in S: i \leq k} G_i \prod_{i \in S: i > k} b_i \right)^* (\widehat{Q}(S))^* \right) \\ &= \sum_{S \subseteq \{1, \dots, m\}} \widehat{Q}(S) (\widehat{Q}(S))^* \\ &\stackrel{(7)}{=} \mathbb{E}_{b_i \sim \mathcal{B}} |Q\{b_i\}|^2. \end{aligned}$$

Then (17) follows. Equation (18) follows from (11), Definition 2.1, and Lemma 2.6. Equalities (19) and (20) follow from Definition 2.1. \square

2.12. Unique Games Conjecture. The Unique Games Conjecture is a commonly assumed conjecture in complexity theory, though its current status is unresolved.

Definition 2.7 (Unique Games). Let $m \in \mathbb{N}$. Let $G = G(\mathcal{S}, \mathcal{W}, \mathcal{E})$ be a bipartite graph with vertex sets \mathcal{S} and \mathcal{W} and edge set $\mathcal{E} \subseteq \mathcal{S} \times \mathcal{W}$. For all $(v, w) \in \mathcal{E}$, let $\pi_{vw}: \{1, \dots, m\} \rightarrow \{1, \dots, m\}$ be a permutation. An instance of the Unique Games problem is $\mathcal{L} = (G(\mathcal{S}, \mathcal{W}, \mathcal{E}), m, \{\pi_{vw}\}_{(v,w) \in \mathcal{E}})$. m is called the alphabet size of \mathcal{L} . A labeling of \mathcal{L} is a function $\eta: \mathcal{S} \cup \mathcal{W} \rightarrow \{1, \dots, m\}$. An edge $(v, w) \in \mathcal{E}$ is satisfied if and only if $\eta(v) = \pi_{vw}(\eta(w))$. The goal of the Unique Games problem is to maximize the fraction of satisfied edges of the labeling, over all such labelings η . We call the maximum possible fraction of satisfied edges $\text{OPT}(\mathcal{L})$.

Definition 2.8 (Unique Games Conjecture, [Kho02, KKMO07]). For every $0 < \beta < \alpha < 1$ there exists $m = m(\alpha, \beta) \in \mathbb{N}$ and a family of Unique Games instances $(\mathcal{L}_n)_{n \geq 1}$ with alphabet size m such that no polynomial time algorithm can distinguish between $\text{OPT}(\mathcal{L}_n) < \beta$ or $\text{OPT}(\mathcal{L}_n) > \alpha$.

3. MAJORIZATION PRINCIPLE

3.1. A noncommutative hypercontractive inequality. One of the main tools used in the proof of our majorization principle is a hypercontractive inequality for noncommutative multilinear polynomials. The inequality bounds the $2K$ -norm, for $K \geq 1$ an integer, of a polynomial Q by the 2-norm of Q . We refer to this inequality as a $(2K, 2)$ hypercontractive inequality; it can be considered as a polynomial generalization of the noncommutative Khintchine inequality between the $2K$ norm and the 2 norm. (The Khintchine inequality corresponds to the polynomial $Q(X_1, \dots, X_m) = X_1$ and $G_1 \sim \mathcal{V}$; see [MJC⁺14, Corollary 7.3] or [DR13].)

Recall the definition of the ensembles \mathcal{G} and \mathcal{V} in Section 2.3.

Theorem 3.1 (($2K, 2$) Hypercontractivity). *Let $K \in \mathbb{N}$. Let Q be a noncommutative multilinear polynomial of degree $d \in \mathbb{N}$, as in (8). Let $G_j \sim \mathcal{G}$, where \mathcal{G} is such that $\|\mathbb{E}(G_1 G_1^*)^K\| \leq c_K$ for some $c_K \geq 1$. Then*

$$\mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} |Q\{G_j\}|^{2K} \leq (2K - 1)^{dK} c_K^d \left(\mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} |Q\{G_j\}|^2 \right)^K.$$

Or, more generally,

$$\mathbb{E}_{G_j \sim \mathcal{G}} \operatorname{Tr} |Q\{G_j\}|^{2K} \leq (2K - 1)^{2dK} c_K^d \left(\mathbb{E}_{G_j \sim \mathcal{G}} \operatorname{Tr} |Q\{G_j\}|^2 \right)^K.$$

Remark 3.2. As mentioned in [MOO10, Theorem 3.13] or [Nel73, Theorem 4], the best possible constant in this hypercontractive inequality is $(2K - 1)^{dK}$ in the case that $G_j \sim \mathcal{V}$ are replaced with $b_i \sim \mathcal{B}$. So, we achieve the optimal constant in this case, since we can use $c_K = 1$ for all $K \in \mathbb{N}$ in this case.

The result that hypercontractivity also holds for the variables $G_1 = b_1 \sim \mathcal{B}$ generalizes a result of [Gro72, Lemma 6.1].

Proof. We begin with the first inequality. It suffices to prove the following hypercontractive estimate: if $\rho \geq 0$ satisfies $\rho \leq (2K - 1)^{-1/2} c_K^{-1/(2K)}$, then

$$\mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} |T_\rho Q\{G_j\}|^{2K} \leq \left(\mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} |Q\{G_j\}|^2 \right)^K. \quad (21)$$

To see that (21) implies the first inequality of the theorem, choose $\rho = (2K - 1)^{-1/2} c_K^{-1/(2K)}$, and observe that

$$\begin{aligned} \mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} |Q\{G_j\}|^{2K} &\stackrel{(14)}{=} \mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} \left| T_\rho \left(\sum_{S \subseteq \{1, \dots, m\}: |S| \leq d} \rho^{-|S|} \widehat{Q}(S) \prod_{i \in S} G_i \right) \right|^{2K} \\ &\stackrel{(21)}{\leq} \left(\mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} \left| \sum_{S \subseteq \{1, \dots, m\}: |S| \leq d} \rho^{-|S|} \widehat{Q}(S) \prod_{i \in S} G_i \right|^2 \right)^K \stackrel{(17) \wedge (7)}{=} \left(\sum_{S \subseteq \{1, \dots, m\}: |S| \leq d} \rho^{-2|S|} \operatorname{Tr} |\widehat{Q}(S)|^2 \right)^K \end{aligned} \quad (22)$$

$$\leq \rho^{-2dK} \left(\sum_{S \subseteq \{1, \dots, m\}: |S| \leq d} \operatorname{Tr} |\widehat{Q}(S)|^2 \right)^K \stackrel{(17) \wedge (7)}{=} (2K - 1)^{dK} c_K^d \left(\mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} |Q\{G_j\}|^2 \right)^K. \quad (23)$$

The proof of (21) is by induction on the number m of variables of Q . If $m = 0$ then there are no variables and the inequality follows from the elementary inequality $\sum_{i=1}^n (\lambda_i(Q))^{2K} \leq (\sum_{i=1}^n (\lambda_i(Q))^2)^K$ applied to the singular values of the (deterministic) matrix Q . To establish

the inductive step, write $Q = R_0 + R_1 X_m$, where R_0, R_1 depend on at most $m - 1$ variables each (for clarity we suppress this dependence from the notation). Note that $T_\rho Q = T_\rho R_0 + \rho(T_\rho R_1) X_m$. We begin with a binomial expansion

$$|T_\rho Q(X_1, \dots, X_m)|^{2K} = \sum_{(a_1, \dots, a_{2K}) \in \{0,1\}^{2K}} \prod_{i=1}^K \rho^{a_{2i-1}} T_\rho R_{a_{2i-1}} X_m^{a_{2i-1}} (X_m^*)^{a_{2i}} \rho^{a_{2i}} (T_\rho R_{a_{2i}})^*. \quad (24)$$

Any term in the sum for which $a_j = 0$ for an odd number of elements $j \in \{1, \dots, 2K\}$ has expectation zero. Applying Hölder's inequality,

$$\mathbb{E}_{G_j \sim \mathcal{V}} \text{Tr} |T_\rho Q|^{2K} \leq \mathbb{E} \text{Tr} |T_\rho R_0|^{2K} + \sum_{\substack{(a_1, \dots, a_{2K}) \in \{0,1\}^{2K}: \\ \frac{1}{2} \sum_{i=1}^{2K} a_i \text{ is a positive integer}}} \prod_{i=1}^{2K} \rho^{a_i} \left[\mathbb{E} \text{Tr} |(T_\rho R_{a_i}) G_m^{a_i}|^{2K} \right]^{\frac{1}{2K}}. \quad (25)$$

Let $(a_1, \dots, a_{2K}) \in \{0,1\}^{2K}$ such that $\ell := \frac{1}{2} \sum_{i=1}^{2K} a_i$ is a positive integer. The number of times the term $\prod_{i=1}^{2K} \rho^{a_i} \left[\mathbb{E} \text{Tr} |(T_\rho R_{a_i}) G_m^{a_i}|^{2K} \right]^{\frac{1}{2K}}$ is repeated in the sum in (25) is $\binom{2K}{2\ell} = \frac{(2K)!}{(2K-2\ell)!(2\ell)!}$. That is, (25) can be rewritten as

$$\mathbb{E}_{G_j \sim \mathcal{V}} \text{Tr} |T_\rho Q|^{2K} \leq \mathbb{E} \text{Tr} |T_\rho R_0|^{2K} + \sum_{\ell=1}^K \rho^{2\ell} \binom{2K}{2\ell} \left[\mathbb{E} \text{Tr} |(T_\rho R_1) G_m|^{2K} \right]^{\frac{\ell}{K}} \left[\mathbb{E} \text{Tr} |T_\rho R_0|^{2K} \right]^{\frac{K-\ell}{K}}. \quad (26)$$

For any $A, B \in M_n(\mathbb{C})$ it holds that $\text{Tr} | |B| A^* A |B| |^K \leq \text{Tr} | |B|^K (A^* A)^K |B|^K |$ (see e.g. [Bha97, Theorem IX.2.10]), hence

$$\text{Tr} (A B B^* A^*)^K = \text{Tr} (A^* A B B^*)^K \leq \text{Tr} ((B B^*)^K (A^* A)^K). \quad (27)$$

Starting from (26) and applying (27) to each of the inner terms,

$$\begin{aligned} \mathbb{E}_{G_j \sim \mathcal{V}} \text{Tr} |T_\rho Q|^{2K} &\leq \mathbb{E} \text{Tr} |T_\rho R_0|^{2K} + \sum_{\ell=1}^K \rho^{2\ell} \binom{2K}{2\ell} c_K^{\ell/K} \left[\mathbb{E} \text{Tr} |T_\rho R_1|^{2K} \right]^{\frac{\ell}{K}} \left[\mathbb{E} \text{Tr} |T_\rho R_0|^{2K} \right]^{\frac{K-\ell}{K}} \\ &\leq [\mathbb{E} \text{Tr} |R_0|^2]^K + \sum_{\ell=1}^K \rho^{2\ell} \binom{2K}{2\ell} c_K^{\ell/K} \left[\mathbb{E} \text{Tr} |R_1|^2 \right]^\ell \left[\mathbb{E} \text{Tr} |R_0|^2 \right]^{K-\ell}, \end{aligned} \quad (28)$$

where the second inequality is obtained by applying the inductive hypothesis. For any odd integer J we denote $J!! = \prod_{i=0}^{(J-1)/2} (J-2i)$, or $J!! = 1$ if $(J-1)/2 < 1$. Now if $1 \leq \ell \leq K$,

$$\begin{aligned} \binom{2K}{2\ell} \binom{K}{\ell}^{-1} &= \frac{(2K)!}{K!} \frac{\ell!}{(2\ell)!} \frac{(K-\ell)!}{(2(K-\ell))!} = \frac{2^K (2K-1)!!}{2^\ell (2\ell-1)!! 2^{K-\ell} (2(K-\ell)-1)!!} \\ &= \frac{(2K-1)!!}{(2\ell-1)!! (2(K-\ell)-1)!!} = \prod_{i=0}^{\ell-1} \frac{2K-2i-1}{2\ell-2i-1} \leq (2K-1)^\ell. \end{aligned}$$

Using this inequality and $0 \leq \rho \leq (2K-1)^{-1/2} c_K^{-1/(2K)}$ we get

$$\rho^{2\ell} \binom{2K}{2\ell} c_K^{\ell/K} \leq \rho^{2\ell} (2K-1)^\ell c_K^{\ell/K} \binom{K}{\ell} \leq \binom{K}{\ell}, \quad \forall 1 \leq \ell \leq K.$$

Applying this inequality to (28),

$$\begin{aligned} \mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} |T_\rho Q|^{2K} &\leq \sum_{\ell=0}^K \binom{K}{\ell} \left[\mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} |R_1|^2 \right]^\ell \left[\mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} |R_0|^2 \right]^{K-\ell} \\ &= \left(\sum_{i=0}^1 \mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} (R_i R_i^*) \right)^K = \left[\mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} (Q Q^*) \right]^K, \end{aligned}$$

where the last equality follows from $\mathbb{E} G_1 G_1^* = I$.

We now prove the second inequality of the Theorem. We repeat the above proof, though now the sum in (25) ranges over all $(a_1, \dots, a_{2K}) \in \{0, 1\}^{2K}$ such that $\sum_{i=1}^{2K} a_i \geq 2$. If $\ell > 0$ satisfies $2\ell + 1 = \sum_{i=1}^{2K} a_i$, then the corresponding term appears $\binom{2K}{2\ell+1}$ times in (25), and we bound this term in (25) with the arithmetic-mean geometric-mean inequality as follows.

$$\begin{aligned} \prod_{i=1}^{2K} \left[\mathbb{E} \operatorname{Tr} |(T_\rho R_{a_i}) G_m^{a_i}|^{2K} \right]^{\frac{1}{2K}} &= \left[\mathbb{E} \operatorname{Tr} |(T_\rho R_1) G_m|^{2K} \right]^{\frac{2\ell+1}{2K}} \left[\mathbb{E} \operatorname{Tr} |T_\rho R_0|^{2K} \right]^{\frac{2K-2\ell-1}{2K}} \\ &\leq \frac{1}{2} \left[\mathbb{E} \operatorname{Tr} |(T_\rho R_1) G_m|^{2K} \right]^{\frac{\ell}{K}} \left[\mathbb{E} \operatorname{Tr} |T_\rho R_0|^{2K} \right]^{\frac{K-\ell}{K}} \\ &\quad + \frac{1}{2} \left[\mathbb{E} \operatorname{Tr} |(T_\rho R_1) G_m|^{2K} \right]^{\frac{\ell+1}{K}} \left[\mathbb{E} \operatorname{Tr} |T_\rho R_0|^{2K} \right]^{\frac{K-\ell-1}{K}}. \end{aligned}$$

Then, as above, we have $\frac{1}{2} \binom{2K}{2\ell+1} \binom{K}{\ell}^{-1} \leq \frac{K-1}{3} (2K-1)^\ell$ and $\frac{1}{2} \binom{2K}{2\ell+1} \binom{K}{\ell+1}^{-1} \leq \frac{2}{3} (2K-1)^\ell$. So, if $0 \leq \rho \leq (2K-1) c_K^{-1/(2K)}$, then $\rho^{2\ell} \left(\binom{2K}{2\ell} + \frac{1}{2} \binom{2K}{2\ell+1} + \frac{1}{2} \binom{2K}{2\ell-1} \right) c_K^{\ell/K} \leq \binom{K}{\ell}$, for all $1 \leq \ell \leq K$, and then the proof concludes as above. \square

Corollary 3.3 ((2K, 2) Hypercontractivity). *Let $K \in \mathbb{N}$. Let Q be a noncommutative multilinear polynomial of degree $d \in \mathbb{N}$, as in (8). Then*

$$\mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} |Q\{G_j\}|^{2K} \leq (2K-1)^{dK} c_K^d (K!)^d \left(\mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} |Q\{G_j\}|^2 \right)^K.$$

$$\mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H_j \sim \mathcal{H}_p}} \operatorname{Tr} |Q^\ell\{G_j H_j\}|^{2K} \leq (2K-1)^{dK} c_K^d (K!)^d \left(\mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H_j \sim \mathcal{H}_p}} \operatorname{Tr} |Q^\ell\{G_j H_j\}|^2 \right)^K.$$

$$\mathbb{E}_{b_j \sim \mathcal{B}} \operatorname{Tr} |Q\{b_j\}|^{2K} \leq (2K-1)^{dK} c_K^d \left(\mathbb{E}_{b_j \sim \mathcal{B}} \operatorname{Tr} |Q\{b_j\}|^2 \right)^K.$$

Proof. The first inequality follows from Theorem 3.1 using [HT99, Corollary 2.8] to show that for $G \sim \mathcal{V}$ and any $\ell \in \mathbb{N}$, $\|\mathbb{E}(GG^*)^\ell\| \leq \ell!$.

To prove the second inequality, we follow the proof of Theorem 3.1 using $\|\mathbb{E}(GG^*)^\ell\| \leq \ell!$ for any $G \sim \mathcal{V} \otimes I$, where the Q used in the proof becomes Q^ℓ . Writing $Q^\ell = R_0^\ell + R_1^\ell X_m$, there are only two required changes. First, the equalities (22) and (23) are justified by

combining (20) with (17) and (7). For example, (22) is justified by

$$\begin{aligned}
& \left(\mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H_j \sim \mathcal{H}_p}} \operatorname{Tr} \left| \sum_{S \subseteq \{1, \dots, m\}: |S| \leq d} \rho^{-|S|} \widehat{Q}^\ell(S) \prod_{i \in S} G_i^\ell H_i \right|^2 \right)^K \\
& \stackrel{(20)}{=} \left(\mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} \left| \sum_{S \subseteq \{1, \dots, m\}: |S| \leq d} \rho^{-|S|} \widehat{Q}(S) \prod_{i \in S} G_i \right|^2 \right)^K \\
& \stackrel{(17) \wedge (7)}{=} \left(\sum_{S \subseteq \{1, \dots, m\}: |S| \leq d} \rho^{-2|S|} \operatorname{Tr} |\widehat{Q}(S)|^2 \right)^K.
\end{aligned}$$

And (23) is justified in the same way. Similarly, the last inequality in the proof is justified as

$$\begin{aligned}
\mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H_j \sim \mathcal{H}_p}} \operatorname{Tr} |T_\rho Q^\ell|^{2K} & \leq \sum_{\ell=0}^K \binom{K}{\ell} \left[\mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H_j \sim \mathcal{H}_p}} \operatorname{Tr} |R_1^\ell|^2 \right]^\ell \left[\mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H_j \sim \mathcal{H}_p}} \operatorname{Tr} |R_0^\ell|^2 \right]^{K-\ell} \\
& \stackrel{(20)}{=} \sum_{\ell=0}^K \binom{K}{\ell} \left[\mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} |R_1|^{2\ell} \right]^\ell \left[\mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} |R_0|^{2(K-\ell)} \right]^{K-\ell} \\
& = \left(\mathbb{E}_{G_j \sim \mathcal{V}} \operatorname{Tr} |Q|^2 \right)^K \\
& \stackrel{(20)}{=} \left(\mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H_j \sim \mathcal{H}_p}} \operatorname{Tr} |Q^\ell|^2 \right)^K.
\end{aligned}$$

The last inequality in the Corollary follows directly from Theorem 3.1. \square

In summary, Q is hypercontractive when we substitute into Q the noncommutative random variables $G_i H_i$. Since Q is also hypercontractive when we substitute into Q commutative random variables distributed uniformly in $\{-1, 1\}$, we get the standard consequence that Q is hypercontractive when we substitute into it a mixture of commutative and noncommutative random variables.

Corollary 3.4 ((2K, 2) Hypercontractivity for mixed inputs). *Let $G_j \sim \mathcal{G}$ be i.i.d. random $n \times n$ matrices and let Q be a noncommutative multilinear polynomial of degree $d \in \mathbb{N}$ such that Q satisfies (2K, 2) hypercontractivity for some $K \in \mathbb{N}$. That is, assume there exists $c_K \geq 1$ such that*

$$\mathbb{E}_{G_j \sim \mathcal{G}} \operatorname{Tr} |Q\{G_j\}|^{2K} \leq (2K - 1)^{2dK} c_K^d \left(\mathbb{E}_{G_j \sim \mathcal{G}} \operatorname{Tr} |Q\{G_j\}|^2 \right)^K.$$

Let $\mathcal{X} = (G_1, \dots, G_k)$. Let $b_j \sim \mathcal{B}$ and let $\mathcal{Y} = (b_{k+1}, \dots, b_m)$. Then

$$\mathbb{E}_{x \sim \mathcal{X}, y \sim \mathcal{Y}} \operatorname{Tr} |Q(x, y)|^{2K} \leq (2K - 1)^{2dK} c_K^d \left(\mathbb{E}_{x \sim \mathcal{X}, y \sim \mathcal{Y}} \operatorname{Tr} |Q(x, y)|^2 \right)^K.$$

Proof. For any $p \geq 1$, let $\|\cdot\|_{p,\mathcal{Z}}$ denote the norm $\|Q\|_{p,\mathcal{Z}} = (\mathbb{E}_{z \sim \mathcal{Z}} \|Q(z)\|_p^p)^{1/p}$.

$$\begin{aligned} \|Q\|_{2K,\mathcal{X} \cup \mathcal{Y}} &= \left\| \sum_{S \subseteq \{1, \dots, m\}} \widehat{Q}(S) \prod_{j \in S: j \leq k} G_j \prod_{j \in S: j > k} b_j \right\|_{2K,\mathcal{X} \cup \mathcal{Y}} \\ &= \left\| \left\| \sum_{S \subseteq \{1, \dots, m\}} \widehat{Q}(S) \prod_{j \in S: j \leq k} G_j \prod_{j \in S: j > k} b_j \right\|_{2K,\mathcal{X}} \right\|_{2K,\mathcal{Y}} \\ &\leq (2K-1)^k c_K^{k/(2K)} \left\| \left\| \sum_{S \subseteq \{1, \dots, m\}} \left(\widehat{Q}(S) \prod_{j \in S: j > k} b_j \right) \prod_{j \in S: j \leq k} G_j \right\|_{2,\mathcal{X}} \right\|_{2K,\mathcal{Y}}, \end{aligned}$$

by Theorem 3.1. Next, using Minkowski's inequality, from the above we get

$$\begin{aligned} \|Q\|_{2K,\mathcal{X} \cup \mathcal{Y}} &\leq (2K-1)^k c_K^{k/(2K)} \left\| \left\| \sum_{S \subseteq \{1, \dots, m\}} \left(\widehat{Q}(S) \prod_{j \in S: j > k} b_j \right) \prod_{j \in S: j \leq k} G_j \right\|_{2K,\mathcal{Y}} \right\|_{2,\mathcal{X}} \\ &= (2K-1)^k c_K^{k/(2K)} \left\| \left\| \sum_{S \subseteq \{1, \dots, m\}} \left(\widehat{Q}(S) \prod_{j \in S: j \leq k} G_j \right) \prod_{j \in S: j > k} b_j \right\|_{2K,\mathcal{Y}} \right\|_{2,\mathcal{X}} \\ &\leq (2K-1)^d c_K^{d/(2K)} \left\| \left\| \sum_{S \subseteq \{1, \dots, m\}} \left(\widehat{Q}(S) \prod_{j \in S: j \leq k} G_j \right) \prod_{j \in S: j > k} b_j \right\|_{2,\mathcal{Y}} \right\|_{2,\mathcal{X}} \\ &= (2K-1)^d c_K^{d/(2K)} \left\| \sum_{S \subseteq \{1, \dots, m\}} \widehat{Q}(S) \prod_{j \in S: j \leq k} G_j \prod_{j \in S: j > k} b_j \right\|_{2,\mathcal{X} \cup \mathcal{Y}} \\ &= (2K-1)^d c_K^{d/(2K)} \|Q\|_{2,\mathcal{X} \cup \mathcal{Y}}, \end{aligned}$$

where the third line is by Theorem 3.1. □

3.2. Majorization principle.

Theorem 3.5 (Noncommutative Majorization Principle for Increasing Test Functions). *Let $Q \in M_n(\mathbb{C})[X_1, \dots, X_m]$ be a noncommutative multilinear polynomial of degree d such that $\mathbb{E}_{b_j \sim \mathcal{B}} \frac{1}{n} \text{Tr}(Q\{b_j\}Q\{b_j\}^*) \leq 1$. Let $\tau := \max_{1 \leq j \leq m} \text{Inf}_j Q$. Let $G_i \sim \mathcal{V} \otimes I$. Assume $\|\mathbb{E}(G_1 G_1^*)^2\| \leq c_2$ and $\|\mathbb{E}(G_1 G_1^*)^3\| \leq c_3$ with $c_2, c_3 \geq 1$.*

Let $\psi: [0, \infty) \rightarrow \mathbb{R}$ be a function with three continuous derivatives such that $\psi'(t) \geq 0$ for all $t \geq 0$. Let $a_2 = \sup_{t \geq 0} |\psi''(t)|$ and $a_3 = \sup_{t \geq 0} |\psi'''(t)|$. Then

$$\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} \psi |Q'\{G_i H_i\}|^2 \leq \mathbb{E}_{b_j \sim \mathcal{B}} \frac{1}{n} \text{Tr} \psi |Q\{b_j\}|^2 + a_3 n^{3/2} (5^6 c_2 c_3)^d \tau^{1/2} + O_{n,m}(a_2 p^{-1/2}).$$

Proof. We show the bound using the Lindeberg replacement method, replacing the m variables $G_j H_j$ by b_j one at a time, starting from the last, for each $j \in \{1, \dots, m\}$. Suppose variables $j+1, j+2, \dots, m$ have already been replaced, and write $Q^t = R + S X_j$ where R, S do not depend on the j^{th} variable.

Any three times continuously differentiable $F: [0, \infty) \rightarrow \mathbb{R}$ has a Taylor expansion

$$F(1) = F(0) + F'(0) + \frac{1}{2} F''(0) + \frac{1}{2} \int_0^1 (1-s)^2 F'''(s) ds. \quad (29)$$

Let $F(t) = \text{Tr}\psi((R + tSX)(R + tSX)^*)$ for $t \in [0, 1]$. Then

$$F(0) = \text{Tr}\psi(RR^*), \quad (30)$$

$$F'(0) = \text{Tr}(\psi'(RR^*)(SXR^* + RX^*S^*)), \quad (31)$$

$$F''(0) = \text{Tr}(\psi'(RR^*)2SXX^*S^* + \psi''(RR^*)(SXR^* + RX^*S^*)^2), \quad (32)$$

$$\begin{aligned} F'''(t) &= \text{Tr}(\psi''((R + tSX)(R + tSX)^*)((SXR^* + RX^*S^*) + 2tSXX^*S^*)(SXX^*S^*)) \\ &\quad + \text{Tr}(\psi''((R + tSX)(R + tSX)^*)2((SXR^* + RX^*S^*) + 2tSXX^*S^*)(SXX^*S^*)) \\ &\quad + \text{Tr}(\psi'''((R + tSX)(R + tSX)^*)2((SXR^* + RX^*S^*) + 2tSXX^*S^*)^3). \end{aligned} \quad (33)$$

For any $t \in [0, 1]$, let $F_1(t) = \text{Tr}\psi((R + tSb_j)(R + tSb_j)^*)$ and $F_2(t) = \text{Tr}\psi((R + tSG_jH_j)(R + tSG_jH_j)^*)$. From (29),

$$\mathbb{E}F_2(1) - \mathbb{E}F_1(1) = \mathbb{E}F_2''(0) - \mathbb{E}F_1''(0) + \mathbb{E}\frac{1}{2}\int_0^1(1-s)^2F_2'''(s)ds - \mathbb{E}\frac{1}{2}\int_0^1(1-s)^2F_1'''(s)ds, \quad (34)$$

where we used that $\mathbb{E}F_2(0) = \mathbb{E}F_1(0)$ and $\mathbb{E}F_2'(0) = \mathbb{E}F_1'(0)$. We bound the two differences on the right-hand side of (34) separately.

For the first, using that $\psi'(RR^*)$ is positive semidefinite the first term can be bounded as

$$\begin{aligned} \mathbb{E}\text{Tr}(\psi'(RR^*)2SG_jH_jH_j^*G_jS^*) &= \mathbb{E}\text{Tr}(\psi'(RR^*)2S\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}S^*) \\ &\leq \mathbb{E}\text{Tr}(\psi'(RR^*)2SS^*) \\ &= \mathbb{E}\text{Tr}(\psi'(RR^*)2Sb_jb_j^*S^*). \end{aligned} \quad (35)$$

The second term in (32) is readily bounded using Corollary 2.5, from which it follows that $\mathbb{E}\|SG_jH_jR^* + RH_j^*G_j^*S^*\| = O_{n,m}(p^{-1/2})$, and $|\psi''(t)| \leq a_2$ for all $t \geq 0$. Combining the two bounds,

$$\mathbb{E}F_2''(0) - \mathbb{E}F_1''(0) \leq O_{n,m}(a_2p^{-1/2}).$$

For the second difference on the right-hand side of (34), there are two terms, corresponding to the first two lines of (33) and the third line respectively. For the first two terms we apply the Cauchy-Schwarz inequality, isolating the last factor SXX^*S^* and using $|\psi''(t)| \leq a_2$ for all $t \geq 0$ to bound them by

$$\begin{aligned} &3a_2\left(\mathbb{E}\text{Tr}((SXX^*S^*)^2)\right)^{1/2}\left(\mathbb{E}\text{Tr}((SXR^* + RX^*S^* + 2tSXX^*S^*)^2)\right)^{1/2} \\ &\leq 3a_2(3^4c_2)^d\left(\mathbb{E}\text{Tr}(SS^*)\right)\left(6\mathbb{E}\text{Tr}(SXR^*RX^*S^*) + 12t^2\mathbb{E}\text{Tr}((SXX^*S^*)^2)\right)^{1/2} \\ &\leq 3a_2(3^4c_2)^d\left(\mathbb{E}\text{Tr}(SS^*)\right)\left(\mathbb{E}\text{Tr}((SXX^*S^*)^2)(36\mathbb{E}\text{Tr}((RR^*)^2 + 12t^2\mathbb{E}\text{Tr}((SXX^*S^*)^2))\right)^{1/4} \\ &\leq 36a_2(3^4c_2)^dn^{1/2}\left(\mathbb{E}\text{Tr}(SS^*)\right)^{3/2}, \end{aligned}$$

where for the first inequality, the first term is bounded using Corollary 3.4 (first using Corollary 3.3 to show that hypercontractivity holds for Q^t and then applying Corollary 3.4 with $Q = Q^t$) and $\mathbb{E}XX^* \leq I$, and the second term is bounded using $(A + B + C)(A + B + C)^* \leq 4(AA^* + BB^* + CC^*)$, the second inequality uses Cauchy-Schwarz, and the last again Corollary 3.4, $\mathbb{E}XX^* \leq I$, and $\mathbb{E}RR^* \leq I$.

Finally we turn to the second term which appears in the expansion of the second difference on the right-hand side of (34) according to (33), corresponding to the third line of (33). Letting $P = SXR^* + RX^*S^* + 2tSXX^*S^*$, the term can be bounded using Hölder's inequality by

$$\begin{aligned} a_3 \mathbb{E} \text{Tr} |P|^3 &= O(a_3) (\mathbb{E} \text{Tr} |RXS^*|^3 + \mathbb{E} \text{Tr} |SXX^*S^*|^3) \\ &= O(a_3) (\mathbb{E} \text{Tr} |SX|^6)^{1/2} \left((\mathbb{E} \text{Tr} |R|^6)^{1/2} + (\mathbb{E} \text{Tr} |SX|^6)^{1/2} \right) \\ &= O(a_3) n^{3/2} (5^6 c_3)^d (\mathbb{E} \text{Tr} |S|^2)^{3/2}, \end{aligned}$$

where the last line uses Corollary 3.4 (applied as above) and $\mathbb{E}SS^* \leq I$, $\mathbb{E}RR^* \leq I$.

Combining all error estimates and using $\mathbb{E} \text{Tr} SS^* = \text{Inf}_j(Q)$ we obtain

$$\begin{aligned} &\mathbb{E} \frac{1}{n} F_2(1) \\ &\leq \mathbb{E} \frac{1}{n} F_1(1) + O_{n,m}(p^{-1/2}) + \mathbb{E} \frac{1}{2n} \int_0^1 (1-s)^2 |F_2'''(s)| ds + \mathbb{E} \frac{1}{2n} \int_0^1 (1-s)^2 |F_1'''(s)| ds \\ &\leq \mathbb{E} \frac{1}{n} F_1(1) + O_{n,m}(a_2 p^{-1/2}) + a_3 n^{1/2} (5^6 c_3)^d (\text{Inf}_j(Q))^{3/2}. \end{aligned} \quad (36)$$

Iterating over all m variables,

$$\begin{aligned} &\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} \psi |Q^i \{G_i H_i\}|^2 - \mathbb{E}_{b_i \sim \mathcal{B}} \frac{1}{n} \text{Tr} \psi |Q\{b_i\}|^2 \\ &\leq a_3 n^{1/2} (5^6 c_2 c_3)^d \left(\max_{1 \leq j \leq m} \text{Inf}_j(Q) \right)^{1/2} \left(\sum_{j=1}^m \text{Inf}_j(Q) \right) + O_{n,m}(p^{-1/2}) \\ &\leq a_3 n^{3/2} (5^6 c_2 c_3)^d \left(\max_{1 \leq j \leq m} \text{Inf}_j(Q) \right)^{1/2} + O_{n,m}(a_2 p^{-1/2}). \end{aligned}$$

□

Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be Lipschitz, so that $\sup_{x \neq y \in \mathbb{R}} \frac{|\psi(x) - \psi(y)|}{|x - y|} \leq 1$. Let $x \in \mathbb{R}$ and let $\phi(x) = e^{-x^2/2} / \sqrt{2\pi}$, $\phi: \mathbb{R} \rightarrow \mathbb{R}$. For any $\lambda > 0$, define $\phi_\lambda(x) = \lambda^{-1} \phi(x/\lambda)$. Define

$$\psi_\lambda(x) = \psi * \phi_\lambda(x) = \int_{\mathbb{R}} \psi(t) \phi_\lambda(x - t) dt. \quad (37)$$

Then $|\psi(x) - \psi_\lambda(x)| < \lambda$ for all $x \in \mathbb{R}$, and $|\frac{d^k}{dx^k} \phi_\lambda(x)| \leq \lambda^{1-k}$ for all $x \in \mathbb{R}$, so that

$$\left| \frac{d^k}{dx^k} \psi_\lambda(x) \right| \leq 3\lambda^{1-k}, \quad \forall x \in \mathbb{R}, \quad \forall 1 \leq k \leq 3.$$

Lemma 3.6. *Let $\lambda > 0$. If ψ is convex, then ψ_λ is convex, and $\psi(x) \leq \psi_\lambda(x)$ for all $x \in \mathbb{R}$.*

Proof. The first property is a standard differentiation argument for convolutions. Since $\psi(x+h) + \psi(x-h) - 2\psi(x) \geq 0$ for all $x, h \in \mathbb{R}$, we also have $\psi_\lambda(x+h) + \psi_\lambda(x-h) - 2\psi_\lambda(x) \geq 0$. The second property follows from Jensen's inequality. □

Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$\psi(t) = \max(0, |t| - 1), \quad \forall t \in \mathbb{R}. \quad (38)$$

Theorem 3.7. Let $Q \in M_n(\mathbb{C})[X_1, \dots, X_m]$ be a noncommutative multilinear polynomial of degree d with $\mathbb{E}_{b_j \sim \mathcal{B}} \frac{1}{n} \text{Tr}(|Q\{b_j\}|^2) \leq 1$, and let $\tau = \max_{i=1, \dots, m} \text{Inf}_i(Q)$. Let $G_i \sim \mathcal{G} \otimes I$. Assume $\|\mathbb{E}(G_1 G_1^*)^2\| \leq c_2$ and $\|\mathbb{E}(G_1 G_1^*)^3\| \leq c_3$ with $c_2, c_3 \geq 1$. Let ψ be as in (38). Then

$$\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} \psi |Q^\iota \{G_i H_i\}|^2 \leq \mathbb{E}_{b_j \sim \mathcal{B}} \frac{1}{n} \text{Tr} \psi |Q\{b_j\}|^2 + n^{1/2} (5^6 c_2 c_3)^d \tau^{1/6} + O_{m,n}(\tau^{-1/3} p^{-1/2}).$$

Proof. Let $\lambda > 0$, and define ψ_λ as in (37). From Lemma 3.6, $\psi_\lambda(x) \geq \psi(x) \geq 0$ for all $x \in \mathbb{R}$. So,

$$\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} \psi |Q^\iota \{G_i H_i\}|^2 \leq \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} \psi_\lambda |Q^\iota \{G_i H_i\}|^2. \quad (39)$$

From Theorem 3.5,

$$\begin{aligned} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} \psi_\lambda |Q^\iota \{G_i H_i\}|^2 &\leq \mathbb{E}_{b_i \sim \mathcal{B}} \frac{1}{n} \text{Tr} \psi_\lambda |Q\{b_i\}|^2 \\ &+ \lambda^{-2} (O(n^{3/2})(5^6 c_2 c_3)^d \tau^{1/2} + O_{n,m}(p^{-1/2})). \end{aligned} \quad (40)$$

Using $\psi_\lambda(x) \leq \psi(x) + \lambda$ for all $x \geq 0$,

$$\frac{1}{n} \text{Tr} \psi_\lambda |Q(\sigma)|^2 \leq \frac{1}{n} \text{Tr} \psi |Q(\sigma)|^2 + \lambda, \quad \forall \sigma \in \{-1, 1\}^m. \quad (41)$$

Combining (39), (40) and (41) completes the proof, with a choice of λ such that $\lambda^3 = \Theta(n^{3/2} \tau^{1/2})$. \square

Recall the definition of T_ρ in (14) and the function $\text{Chop} : \mathbb{R} \rightarrow \mathbb{R}$, $\text{Chop}(t) = t$ if $|t| \leq 1$, $\text{Chop}(t) = 1$ if $t \geq 1$, and $\text{Chop}(t) = -1$ if $t \leq -1$.

Corollary 3.8 (Smoothed Version of Theorem 3.7). Suppose $f : \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$ with $\|f(\sigma)\| \leq 1$ for all $\sigma \in \{-1, 1\}^m$. Let $0 < \rho < 1$ and let $\tau := \max_{i=1, \dots, m} \text{Inf}_i f$. Assume $\|\mathbb{E}(G_1 G_1^*)^2\| \leq c_2$ and $\|\mathbb{E}(G_1 G_1^*)^3\| \leq c_3$ for some $c_2, c_3 \geq 1$. Then $\|Q_f\|_{L_2, \mathcal{G}} \leq 1$ and

$$\frac{1}{n} \|T_\rho Q_f^\iota - \text{Chop} T_\rho Q_f^\iota\|_{L_2, \mathcal{G}}^2 \leq 10 n^{1/2} \tau^{\frac{1-\rho}{30(c_2 c_3)}} + O_{m,n}(\tau^{-1/3} p^{-1/2}). \quad (42)$$

Proof. Using the elementary inequality $[\max(0, t-1)]^2 \leq \psi(t^2)$ for all $t \geq 0$, where ψ is defined in (38), applied to the singular values of $T_\rho Q_f^\iota$,

$$\frac{1}{n} \|T_\rho Q_f^\iota - \text{Chop} T_\rho Q_f^\iota\|_{L_2, \mathcal{G}}^2 \leq \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} \psi |T_\rho Q_f^\iota \{G_i H_i\}|^2. \quad (43)$$

We first apply Theorem 3.7 to $P_{\leq d}(T_\rho Q_f^\iota)$, where $d \in \mathbb{N}$ is to be determined later. Since $0 < \rho < 1$, (14), (11) imply that

$$\max_{i=1, \dots, m} \text{Inf}_i P_{\leq d} T_\rho Q_f \leq \max_{i=1, \dots, m} \text{Inf}_i P_{\leq d} Q_f \leq \max_{i=1, \dots, m} \text{Inf}_i Q_f,$$

and we get by Theorem 3.7 and (18)

$$\begin{aligned} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} \psi |P_{\leq d} T_\rho Q_f^\iota \{G_i H_i\}|^2 &\leq \mathbb{E}_{b_i \sim \mathcal{B}} \frac{1}{n} \text{Tr} \psi |P_{\leq d} T_\rho Q_f \{b_i\}|^2 \\ &+ n^{1/2} (5^6 c_2 c_3)^d \tau^{1/6} + O_{m,n}(\tau^{-1/3} p^{-1/2}). \end{aligned} \quad (44)$$

For any $a, b \in \mathbb{R}$, $|\psi((a+b)^2) - \psi(a^2)| \leq 2|a||b| + 2|b|^2$ follows by $|\psi((a+b)^2) - \psi(a^2)| \leq |b| \max_{t \in [a, a+b]} \left| \frac{d}{dt} \psi(t^2) \right| \leq |b| 2(|a| + |b|)$. Combining with the Cauchy-Schwarz inequality,

$$\begin{aligned} \text{Tr} \psi |T_\rho Q_f^t|^2 - \text{Tr} \psi |P_{\leq d} T_\rho Q_f^t|^2 &= \text{Tr} \psi |T_\rho P_{\leq d} Q_f^t + T_\rho P_{> d} Q_f^t|^2 - \text{Tr} \psi |T_\rho P_{\leq d} Q_f^t|^2 \\ &\leq 2(\text{Tr} |T_\rho P_{\leq d} Q_f^t|^2)^{1/2} (\text{Tr} |T_\rho P_{> d} Q_f^t|^2)^{1/2} + 2 \text{Tr} |T_\rho P_{> d} Q_f^t|^2. \end{aligned}$$

Taking expectation values and using (20), (17) and (14), which imply that

$$\begin{aligned} \mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H_j \sim \mathcal{H}_p}} \text{Tr} |T_\rho P_{> d} Q_f^t|^2 &= \mathbb{E}_{G_j \sim \mathcal{G}} \text{Tr} |T_\rho P_{> d} Q_f^t|^2 \\ &= \mathbb{E}_{b_j \sim \mathcal{B}} \text{Tr} |T_\rho P_{> d} Q_f^t|^2 \\ &= \sum_{S \subseteq \{1, \dots, m\}: |S| > d} \rho^{2|S|} \text{Tr}(\widehat{f}(S)(\widehat{f}(S))^*) \\ &\leq \rho^{2d} \sum_{S \subseteq \{1, \dots, m\}} \text{Tr}(\widehat{f}(S)(\widehat{f}(S))^*) \\ &\leq n \rho^{2d}, \end{aligned}$$

we get

$$\frac{1}{n} \left| \mathbb{E}_{b_i \sim \mathcal{B}} \left(\text{Tr} \psi |T_\rho Q_f^t \{b_i\}|^2 - \text{Tr} \psi |P_{\leq d} T_\rho Q_f^t \{b_i\}|^2 \right) \right| \leq 4\rho^d, \quad (45)$$

$$\frac{1}{n} \left| \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \left(\text{Tr} \psi |T_\rho Q_f^t \{G_i H_i\}|^2 - \text{Tr} \psi |P_{\leq d} T_\rho Q_f^t \{G_i H_i\}|^2 \right) \right| \leq 4\rho^d. \quad (46)$$

Using $t \leq |t|$ for any $t \in \mathbb{R}$,

$$\begin{aligned} &\frac{1}{n} \left(\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} \psi |T_\rho Q_f^t \{G_i H_i\}|^2 - \mathbb{E}_{b_i \sim \mathcal{B}} \text{Tr} \psi |T_\rho Q_f^t \{b_i\}|^2 \right) \\ &\leq \frac{1}{n} \left| \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} \psi |T_\rho Q_f^t \{G_i H_i\}|^2 - \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} \psi |T_\rho P_{\leq d} Q_f^t \{G_i H_i\}|^2 \right| \\ &\quad + \frac{1}{n} \left(\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} \psi |T_\rho P_{\leq d} Q_f^t \{G_i H_i\}|^2 - \mathbb{E}_{b_i \sim \mathcal{B}} \text{Tr} \psi |T_\rho P_{\leq d} Q_f^t \{b_i\}|^2 \right) \\ &\quad + \frac{1}{n} \left| \mathbb{E}_{b_i \sim \mathcal{B}} \text{Tr} \psi |T_\rho P_{\leq d} Q_f^t \{b_i\}|^2 - \mathbb{E}_{b_i \sim \mathcal{B}} \text{Tr} \psi |T_\rho Q_f^t \{b_i\}|^2 \right| \\ &\leq n^{1/2} (5^6 c_2 c_3)^d \tau^{1/6} + 8\rho^d + O_{m,n}(\tau^{-1/3} p^{-1/2}), \end{aligned} \quad (47)$$

where the last inequality uses (46) to bound the first term, (44) for the second and (45) for the third. From (13) and (14) we get $\|T_\rho Q_f(\sigma)\| \leq 1$ for all $\sigma \in \{-1, 1\}^m$, so by definition of ψ we have $\mathbb{E}_{b_i \sim \mathcal{B}} \text{Tr} \psi |T_\rho Q_f^t \{b_i\}|^2 = 0$. Combining (43) with (47) and choosing $d = \min(\max(1, -\frac{\log(\tau)}{30(c_2 c_3)}), m)$ completes the proof (using $-\log \rho \geq 1 - \rho$ for all $0 < \rho < 1$). \square

3.3. Moment Majorization. Theorem 3.5 implies that the even moments of a noncommutative multilinear polynomial follow a majorization principle. Although we will not make use of Theorem 3.9 for the applications in this paper, we include it as the statement could be of independent interest; the theorem has analogues in both the commutative [MOO10] and free probability settings [DN14].

Theorem 3.9 (Noncommutative Majorization Principle for $2K^{\text{th}}$ Moments). *Let Q be a noncommutative multilinear polynomial of degree d in m variables, as in (8). Suppose $\|Q(\sigma)\| \leq 1$ for all $\sigma \in \{-1, 1\}^m$. Let $p > n$, and let Q^t be the zero-padded extension of Q , as defined in Definition 2.1. Let $\tau := \max_{i=1, \dots, m} \text{Inf}_i Q$. Let $G_i \sim \mathcal{G}$. Assume that $\|\mathbb{E}(G_1 G_1^*)^K\| \leq c_K$, for some $K \in \mathbb{N}$ and $c_K \geq 1$. Then*

$$\begin{aligned} & \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} |Q^t\{G_i H_i\}|^{2K} \\ & \leq \mathbb{E}_{b_i \sim \mathcal{B}} \frac{1}{n} \text{Tr} |Q\{b_i\}|^{2K} + K^3 (2K-1)^{2dK} c_K^d n^{2K} \tau^{1/4} + O_{m,n}(p^{-1/2} K^2 \tau^{-1/4}). \end{aligned} \quad (48)$$

Proof. We begin with an upper tail estimate for Q^t . From Markov's inequality,

$$\mathbb{P}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} (\text{Tr} |Q^t|^2 > t) = \mathbb{P}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} ((\text{Tr} |Q^t|^2)^K > t^K) \leq t^{-K} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} [\text{Tr} |Q^t|^2]^K. \quad (49)$$

Since $Q^t = I^t Q^t$ (where here I denotes the $n \times n$ identity matrix), Hölder's inequality implies

$$\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} (\text{Tr} |Q^t|^2)^K = \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} (\text{Tr} |I^t Q^t|^2)^K \leq n^{K-1} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} |Q^t|^{2K}. \quad (50)$$

Combining (49) and (50) and applying Theorem 3.1,

$$\begin{aligned} \mathbb{P}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} (\text{Tr} |Q^t|^2 > t) & \leq t^{-K} n^{K-1} (2K-1)^{2dK} c_K^d \left(\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} |Q^t|^2 \right)^K \\ & \leq t^{-K} n^{2K-1} (2K-1)^{2dK} c_K^d. \end{aligned} \quad (51)$$

Let $s > 0$ be a constant to be fixed later. Define $\psi: [0, \infty) \rightarrow [0, \infty)$ so that $\psi(t) = t^K$ for any $0 \leq t \leq s$, and ψ is linear with slope Ks^{K-1} on (s, ∞) . It is possible to construct such a ψ with all three derivatives bounded, and in particular the third derivative bounded by some $a_3 \leq K^3 s^{K-3}$ on the interval $[s, s+1]$. From Theorem 3.5,

$$\begin{aligned} & \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} \psi |Q^t\{G_i H_i\}|^2 \\ & \leq \mathbb{E}_{b_i \sim \mathcal{B}} \frac{1}{n} \text{Tr} \psi |Q\{b_i\}|^2 + K^3 s^{K-3} n^{3/2} (5^6 c_2 c_3)^d \tau^{1/2} + O_{n,m}(p^{-1/2} K^2 s^{K-2}). \end{aligned} \quad (52)$$

Finally, defining $a = K^3 s^{K-3} n^{3/2} (5^6 c_2 c_3)^{d\tau^{1/2}} + O_{n,m}(p^{-1/2} K^2 s^{K-2})$ and letting D be the event that $\text{Tr}\psi |Q^\iota\{G_i H_i\}|^2 \leq s$, we have

$$\begin{aligned} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} |Q^\iota\{G_i H_i\}|^{2K} &= \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} |Q^\iota\{G_i H_i\}|^{2K} \cdot 1_D + \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} |Q^\iota\{G_i H_i\}|^{2K} \cdot 1_{D^c} \\ &= \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr}\psi |Q^\iota\{G_i H_i\}|^2 + \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} |Q^\iota\{G_i H_i\}|^{2K} \cdot 1_{D^c}. \end{aligned}$$

Using (52) to bound the first term and Cauchy-Schwarz for the second, the above can be bounded as

$$\begin{aligned} &\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} |Q^\iota\{G_i H_i\}|^{2K} \\ &\leq \mathbb{E}_{b_i \sim \mathcal{B}} \frac{1}{n} \text{Tr}\psi |Q\{b_i\}|^2 + a + \left(\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \left(\frac{1}{n} \text{Tr} |Q^\iota\{G_i H_i\}|^{2K} \right)^2 \right)^{1/2} \left(\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} 1_{D^c} \right)^{1/2} \\ &\leq \mathbb{E}_{b_i \sim \mathcal{B}} \frac{1}{n} \text{Tr}\psi |Q\{b_i\}|^2 + a + \left(\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} |Q^\iota\{G_i H_i\}|^{4K} \right)^{1/2} \left(\mathbb{P}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} (\text{Tr} |Q^\iota|^2 > s) \right)^{1/2} \\ &\leq \mathbb{E}_{b_i \sim \mathcal{B}} \frac{1}{n} \text{Tr}\psi |Q\{b_i\}|^2 + a + (2K-1)^{2dK} c_K^d \left(\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} |Q^\iota\{G_i H_i\}|^2 \right)^K s^{-K/2} n^{K-1}, \end{aligned}$$

using Theorem 3.1 to bound the first term inside a square root, and (51) for the second. Finally, using Lemma 2.6 and $\|Q(\sigma)\| \leq 1$ for all $\sigma \in \{-1, 1\}^m$, $\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} |Q^\iota\{G_i H_i\}|^2 \leq 1$.

Choosing $s = \tau^{-1/(4K)}$ finishes the proof. \square

4. DICTATORSHIP TESTING

Fix integers $m, n, N \in \mathbb{N}$, $M \in M_{n \times n}(\mathbb{C})$ and $V \in \mathcal{U}_n(\mathbb{C}^N)$. Given $f: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$, $f = \sum_{S \subseteq \{1, \dots, m\}} \widehat{f}(S) W_S$, define $\mathfrak{B}(f): \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$ by

$$\mathfrak{B}(f) := \sum_{S \subseteq \{1, \dots, m\}: |S|=1} \text{Tr}_2 \left((V \odot V) M (I \otimes \overline{\widehat{f}(S)}) \right) W_S, \quad (53)$$

where the partial trace Tr_2 is defined for any $X = \sum_i A_i \otimes B_i \in M_{n \times n}(\mathbb{C})$ as $\text{Tr}_2(X) = \sum_i A_i \text{Tr}(B_i)$ and for any $U, V \in M_n(\mathbb{C}^N)$, $U \odot V$ is as defined in Remark 1.11. For any $f, h: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$ let

$$\text{OBJ}(f, h) := 2^{-m} \sum_{\sigma \in \{-1, 1\}^m} \text{Tr}(f(\sigma) \mathfrak{B}(h)(\sigma)), \quad \text{OBJ}(f) := \text{OBJ}(f, f). \quad (54)$$

Using the identity

$$\text{Tr}(A_1(C \otimes C')D_1) = \text{Tr}(C \text{Tr}_2[D_1 A_1(I \otimes C')])$$

valid for any $C, C' \in M_n(\mathbb{C})$ and $A_1, D_1 \in M_{n \times n}(\mathbb{C})$, we can rewrite

$$\text{OBJ}(f, h) = \sum_{S \subseteq \{1, \dots, m\}: |S|=1} \text{Tr}((V \odot V) M (\widehat{f}(S) \otimes \overline{\widehat{h}(S)})). \quad (55)$$

We interpret $\text{OBJ}(f, h)$ as a “dictatorship test”, where dictators are functions $f: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$ such that there exists $i \in \{1, \dots, m\}$ such that for all $\sigma \in \{-1, 1\}^m$, $f(\sigma) = \sigma_i I_n$. The main lemmas of this section state the completeness and soundness properties of this test.

Lemma 4.1 (Completeness). *Let $f: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$ be a dictator. Then*

$$\text{OBJ}(f) = \text{Tr}((V \odot V)M).$$

Proof. Follows directly from (55). □

Lemma 4.2 (Soundness). *Assume $M \in M_{n \times n}(\mathbb{C})$ is PSD. For any $\varepsilon > 0$ there is $\tau = (\varepsilon/n)^{O(\varepsilon^{-1})}$ such that for any $f: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$ such that $\|f(\sigma)\| \leq 1$ for all $\sigma \in \{-1, 1\}^m$ and $\max_{i=1, \dots, m} \text{Inf}_i f \leq \tau$ it holds that*

$$\text{OBJ}(f) \leq (1 + \varepsilon) \sup_{R, Z \in \mathcal{U}_n} \text{Tr}(M \cdot (R \otimes \bar{Z})).$$

The proof of Lemma 4.2 involves the introduction of a Gaussian analogue of (54). For any $f, h: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$ and $p \geq n$, define

$$C_p(f, h) := \mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H_j, J_j \sim \mathcal{H}_p}} \sum_{i=1}^m \text{Tr} \left(M^i \cdot (\mathcal{P}_i Q_f^i \{G_j H_j\} \otimes \overline{\mathcal{P}_i Q_h^i \{G_j J_j\}}) \cdot (H_i^* \otimes \bar{J}_i^*) \right), \quad (56)$$

where the coordinate projection \mathcal{P}_i is defined in (15). For any $0 < \tau < 1$ and $p \geq n$, define

$$C_{\tau, p} := \sup_{\substack{f: \{-1, 1\}^m \rightarrow \mathcal{U}_n \\ \max_{i=1, \dots, m} \text{Inf}_i f \leq \tau}} C_p(f, f). \quad (57)$$

The following lemma equates the two quadratic forms $\text{OBJ}(f, h)$ and $C_p(f, h)$.

Lemma 4.3. *For any $f, h: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$ and $p \geq n$,*

$$\text{OBJ}(f, h) = C_p(f, h).$$

Proof. From the definition (56), $C_p(f, h)$ equals

$$\begin{aligned} & \mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H_j, J_j \sim \mathcal{H}_p}} \sum_{i=1}^m \text{Tr} \left(M^i \cdot (\mathcal{P}_i Q_f^i \{G_j H_j\} \otimes \overline{\mathcal{P}_i Q_h^i \{G_j J_j\}}) \cdot (H_i^* \otimes \bar{J}_i^*) \right) \\ &= \text{Tr} \left(\sum_{S \subseteq \{1, \dots, m\}: |S|=1} \iota(M) (\iota(\widehat{f}(S)) \otimes \overline{\iota(\widehat{h}(S))}) \left(\sum_{j=1}^N V_j \otimes I \otimes \bar{V}_j \otimes I \right) \right) \\ &= \text{Tr} \left(\sum_{S \subseteq \{1, \dots, m\}: |S|=1} \iota(M) (\iota(\widehat{f}(S)) \otimes \overline{\iota(\widehat{h}(S))}) (\iota(V) \odot \iota(V)) \right) \\ &= \text{Tr} \left(\sum_{S \subseteq \{1, \dots, m\}: |S|=1} M(\widehat{f}(S) \otimes \overline{\widehat{h}(S)}) (V \odot V) \right) \\ &= \text{OBJ}(f, h), \end{aligned}$$

where the first equality follows since the expectation over H_j, J_j itself projects onto the linear terms of Q_f, Q_h , and the second equality follows since, if $M = \sum_{k=1}^N C_k \otimes D_k$ with $C_k, D_k \in M_n(\mathbb{C})$ for all $1 \leq k \leq N$, then

$$\iota(M) = \sum_{k=1}^N \iota(C_k) \otimes \iota(D_k) = \sum_{k=1}^N C_k \otimes \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} D_k \otimes \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Finally, the last equality follows from (55). \square

The motivation for introducing $C_p(f, h)$ is the following. On the one hand, Lemma 4.1 tells us the value of $C_p(f, f)$ when f is a dictatorship function. On the other hand, when f has low influences and $\|f(\sigma)\| \leq 1$ for all $\sigma \in \{-1, 1\}^m$, we will show that Q_f with random matrix inputs typically has operator norm bounded by 1. This will let us relate $C_p(f, h)$ to the right-hand side of the inequality stated in Lemma 4.2.

Based on Lemma 4.3, in order to prove Lemma 4.2 it suffices to establish the following.

Lemma 4.4. *Assume $M \in M_{n \times n}(\mathbb{C})$ is PSD. For any $\varepsilon > 0$, for sufficiently small $\tau = (\varepsilon/n)^{O(\varepsilon^{-1})}$ and large enough p (depending on τ, ε, n, m),*

$$C_{\tau, p} \leq (1 + \varepsilon) \sup_{R, Z \in \mathcal{U}_n} \text{Tr}(M \cdot (R \otimes \bar{Z})) \quad (58)$$

Proof. The proof is based on the majorization principle developed in Section 3. To apply the principle, we introduce the following smoothed, truncated analogue of $C_p(f, h)$, for any $1/2 < \rho < 1$,

$$\tilde{C}_{p, \rho}(f, h) := \mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H_j, J_j \sim \mathcal{H}_p}} \sum_{i=1}^m \text{Tr} \left(M^\iota \cdot (\mathcal{P}_i \text{Chop} T_\rho Q_f^\iota \{G_j H_j\} \otimes \overline{\mathcal{P}_i \text{Chop} T_\rho Q_h^\iota \{G_j J_j\}}) \cdot (H_i^* \otimes \bar{J}_i^*) \right). \quad (59)$$

Using (1) from Lemma 2.3,

$$\begin{aligned} \tilde{C}_{p, \rho}(f, f) &\leq \mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H, J \sim \mathcal{H}_p}} \text{Tr} \left(M^\iota \cdot (\text{Chop} T_\rho Q_f^\iota \{G_j H\} \otimes \overline{\text{Chop} T_\rho Q_f^\iota \{G_j H\}}) \cdot (H^* \otimes \bar{J}^*) \right) \\ &\leq \sup_{R, Z \in \mathcal{U}_n} \text{Tr}(M \cdot (R \otimes \bar{Z})), \end{aligned} \quad (60)$$

where the second inequality follows since $\text{Chop} T_\rho Q_f^\iota$ has operator norm at most 1. To conclude the proof of the lemma it will suffice to show that, for appropriate τ, ρ and p ,

$$\sup_{\substack{f: \{-1, 1\}^m \rightarrow \mathcal{U}_n \\ \max_{i=1, \dots, m} \text{Inf}_i f \leq \tau}} (C_p(f, f) - \tilde{C}_{p, \rho}(f, f)) \leq \varepsilon \sup_{R, Z \in \mathcal{U}_n} \text{Tr}(M \cdot (R \otimes \bar{Z})). \quad (61)$$

To alleviate notation write

$$A_i = \mathcal{P}_i T_\rho Q_f^\iota \{G_j H_j\}, \quad \tilde{A}_i = \mathcal{P}_i \text{Chop} T_\rho Q_f^\iota \{G_j H_j\}$$

and similarly

$$B_i = \mathcal{P}_i T_\rho Q_f^\iota \{G_j J_j\}, \quad \tilde{B}_i = \mathcal{P}_i \text{Chop} T_\rho Q_f^\iota \{G_j J_j\}.$$

Then

$$\begin{aligned}
C_p(f, f) &= \mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H_j, J_j \sim \mathcal{H}_p}} \sum_{i=1}^m \text{Tr} \left(M^\iota \cdot (\mathcal{P}_i Q_f^\iota \{G_j H_j\} \otimes \overline{\mathcal{P}_i Q_f^\iota \{G_j J_j\}}) \cdot (H_i^* \otimes \overline{J_i^*}) \right) \\
&= \rho^{-2} \mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H_j, J_j \sim \mathcal{H}_p}} \sum_{i=1}^m \text{Tr} (M^\iota \cdot (A_i \otimes \overline{B_i}) \cdot (H_i^* \otimes \overline{J_i^*})) \\
&= \rho^{-2} \tilde{C}_{p,\rho}(f, f) + \rho^{-2} \sum_{i=1}^m \mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H_j, J_j \sim \mathcal{H}_p}} \text{Tr} (M^\iota \cdot ((A_i - \tilde{A}_i) \otimes \overline{\tilde{B}_i}) \cdot (H_i^* \otimes \overline{J_i^*})) \\
&\quad + \rho^{-2} \sum_{i=1}^m \mathbb{E}_{\substack{G_j \sim \mathcal{V} \otimes I \\ H_j, J_j \sim \mathcal{H}_p}} \text{Tr} (M^\iota \cdot (A_i \otimes \overline{(B_i - \tilde{B}_i)}) \cdot (H_i^* \otimes \overline{J_i^*})), \tag{62}
\end{aligned}$$

where the second equality uses that \mathcal{P}_i projects on the linear part of the multilinear polynomial Q_f^ι , on which T_ρ amounts to multiplication by ρ^{-1} . Interpreting $(A_i - \tilde{A}_i)_{i=1, \dots, m}$ as vector-valued matrices and using (2) from Lemma 2.3,

$$\begin{aligned}
\left\| \sum_{i=1}^m A_i - \tilde{A}_i \right\|_{L_2, \mathcal{V}}^2 &\leq \|T_\rho Q_f^\iota \{G_j H_j\} - \text{Chop} T_\rho Q_f^\iota \{G_j H_j\}\|_{L_2, \mathcal{V}}^2 \\
&\leq 10n^{3/2} \tau^{\frac{1-p}{500}} + O_{m,n}(\tau^{-1/3} p^{-1/2}),
\end{aligned}$$

where the second inequality follows from Corollary 3.8. A similar bound holds for the terms involving $B_i - \tilde{B}_i$, so that from (62) we get

$$C_p(f, f) \leq \rho^{-2} \tilde{C}_{p,\rho}(f, f) + (20n^{3/2} \tau^{\frac{1-p}{500}} + O_{m,n}(\tau^{-1/3} p^{-1/2})) \sup_{R, Z \in \mathcal{U}_p(\mathbb{C}^N)} \text{Tr}(M^\iota \cdot (R \odot Z)). \tag{63}$$

First we take ρ close enough to 1 that $\rho^{-2} \leq (1 + \varepsilon/2)$. Next, bounding the supremum in the right-hand side of (63) using the noncommutative Grothendieck inequality (4), choosing τ small enough as a function of n, ρ and ε so that $20n^{3/2} \tau^{\frac{1-p}{500}} \leq \varepsilon/8$, and finally p large enough so that $O_{m,n}(\tau^{-1/3} p^{-1/2}) \leq \varepsilon/8$ as well, using (60) the right-hand side of (63) is at most $\tilde{C}_{p,\rho}(f, f) + \varepsilon \sup_{R, Z \in \mathcal{U}_n} \text{Tr}(M \cdot (R \otimes \overline{Z}))$. So, taking the supremum over suitable f in (63) proves (61), proving the lemma. \square

5. UNIQUE GAMES HARDNESS

In this section we prove Theorem 1.12, implementing the dictatorship vs. low influences machinery using the tools introduced in Section 4. Let $\mathcal{L}(G(\mathcal{S}, \mathcal{W}, \mathcal{E}), m, \{\pi_{vw}\}_{(v,w) \in \mathcal{E}})$ be a Unique Games instance as in Definition 2.7. Let $v \in \mathcal{S}$. Let $F: \mathcal{W} \times \{-1, 1\}^m \rightarrow \mathcal{U}_n$. Define $F_w(\sigma) := F(w, \sigma)$, for all $w \in \mathcal{W}$ and for all $\sigma = (\sigma_1, \dots, \sigma_m) \in \{-1, 1\}^m$. For each $(v, w) \in \mathcal{E}$, since $\pi_{vw}: \{1, \dots, m\} \rightarrow \{1, \dots, m\}$ is a bijection we can define

$$F_w \circ \pi_{vw}(\sigma) := F_w(\sigma_{\pi_{vw}(1)}, \dots, \sigma_{\pi_{vw}(m)}), \quad \forall \sigma \in \{-1, 1\}^m.$$

For any $v \in \mathcal{S}$ define $F_v: \{-1, 1\}^m \rightarrow \mathcal{U}_n$ by

$$F_v(\sigma) := \mathbb{E}_{(v,w) \in \mathcal{E}} (F_w \circ \pi_{vw}(\sigma)), \quad \forall \sigma \in \{-1, 1\}^m.$$

Unless otherwise stated, expectations and probabilities involving \mathcal{S}, \mathcal{W} and \mathcal{E} will always be taken with respect to the uniform distribution on these sets.

Let $M \in M_{n \times n}(\mathbb{C})$ be PSD, and $V \in \mathcal{U}_n(\mathbb{C}^N)$ such that

$$\mathrm{Tr}(M \cdot (V \odot V)) = \lambda_1 := \sup_{W, Z \in \mathcal{U}_n(\mathbb{C}^N)} |\mathrm{Tr}(M \cdot (W \odot Z))|. \quad (64)$$

When $M \in M_{n \times n}(\mathbb{C})$ is PSD the NCGI becomes simpler in the following sense.

Lemma 5.1. *Let $M \in M_{n \times n}(\mathbb{C})$ be PSD and let \mathcal{D} be a set of matrices. Then*

$$\sup_{X, Y \in \mathcal{D} \times \mathcal{D}} |\mathrm{Tr}(M(X \otimes \bar{Y}))| = \sup_{X \in \mathcal{D}} \mathrm{Tr}(M(X \otimes \bar{X})).$$

Proof. Since M is PSD, by definition it can be written as $M = \sum_{i=1}^N M_i \otimes \bar{M}_i \in M_{n \times n}(\mathbb{C})$ for some $M_1, \dots, M_N \in M_n(\mathbb{C})$. Let $X, Y \in \mathcal{D}$. Using the Cauchy-Schwarz inequality,

$$\begin{aligned} |\mathrm{Tr}(M(X \otimes \bar{Y}))| &= \left| \sum_{i=1}^N \mathrm{Tr}(M_i X) \overline{\mathrm{Tr}(M_i Y)} \right| \\ &\leq \left(\sum_{i=1}^N |\mathrm{Tr}(M_i X)|^2 \right)^{1/2} \left(\sum_{i=1}^N |\mathrm{Tr}(M_i Y)|^2 \right)^{1/2}, \end{aligned}$$

with equality if $X = Y$. □

Let $0 < \rho < 1$. We investigate the quantity

$$\mathrm{OPT}_{\rho, \mathcal{L}}(M) := \sup_{F: \mathcal{W} \times \{-1, 1\}^m \rightarrow \mathcal{U}_n} \mathbb{E}_{v \in \mathcal{S}} [\mathrm{OBJ}(T_\rho F_v)], \quad (65)$$

where $\mathrm{OBJ}(f)$ is defined in (54).

Lemma 5.2 (Completeness). *Suppose the Unique Games instance \mathcal{L} has an almost satisfying labeling, i.e. $(1 - \varepsilon)$ fraction of the labels are satisfied. Then $\forall 0 < \rho < 1$,*

$$\mathrm{OPT}_{\rho, \mathcal{L}}(M) \geq (1 - \varepsilon)\rho\lambda_1.$$

Proof. Let $\eta: \mathcal{S} \cup \mathcal{W} \rightarrow \{1, \dots, m\}$ be a labeling such that, for at least a $1 - \varepsilon$ fraction of edges $(v, w) \in \mathcal{E}$,

$$\pi_{vw}(\eta(v)) = \eta(w).$$

For each $i \in \{1, \dots, m\}$, let $f^{dict, i}(\sigma) := \sigma_i I$, for all $\sigma \in \{-1, 1\}^m$. Define $F: \mathcal{W} \times \{-1, 1\}^m \rightarrow \mathcal{U}_n$ so that, for every $w \in \mathcal{W}$, $F_w: \{-1, 1\}^m \rightarrow \mathcal{U}_n$ satisfies $F_w := f^{dict, \eta(w)}$. Note that $f^{dict, j} \circ \pi = f^{dict, \pi(j)}$ for any $j \in \{1, \dots, m\}$. To see this, assume that $\sigma_{\pi(j)} = \ell$. Then $f^{dict, \pi(j)}(\sigma) = \ell 1_{n \times n}$ since $\sigma_{\pi(j)} = \ell$. Also, $(f^{dict, j} \circ \pi)(\sigma) = f^{dict, j}(\sigma_{\pi(1)}, \dots, \sigma_{\pi(n)}) = \ell 1_{n \times n}$, since $\sigma_{\pi(j)} = \ell$. Therefore, if $\pi_{vw}(\eta(w)) = \eta(v)$, then $F_w \circ \pi_{vw} = f^{dict, \pi_{vw}\eta(w)} = f^{dict, \eta(v)}$, and by Lemma 4.1,

$$\mathrm{OBJ}(T_\rho F_w \circ \pi_{vw}) = \rho\lambda_1.$$

If $\pi_{vw}(\eta(w)) \neq \eta(v)$, then since M is PSD, $|\mathrm{OBJ}(T_\rho F_w \circ \pi_{vw})| \geq 0$. Therefore, $\mathrm{OPT}_{\rho, \mathcal{L}}(M)$ is at least $(1 - \varepsilon)\rho\lambda_1$. □

Lemma 5.3. *Let $0 < \tau, \rho < 1$ and suppose that there exists $\varepsilon > 0$ such that $\mathrm{OPT}_{\rho, \mathcal{L}}(M) > C_{\tau, \rho} + 2\varepsilon$. Then the Unique Games instance \mathcal{L} has a labeling satisfying at least an $\varepsilon\tau^2(1 - \rho)n^{-2}\lambda_1^{-1}/4$ fraction of its edges.*

Proof. Assume that $\text{OPT}_{\rho, \mathcal{L}}(M) > C_{\tau, p} + 2\varepsilon$, and let F be such that²

$$\mathbb{E}_{v \in \mathcal{S}}[\text{OBJ}(T_\rho F_v)] = \text{OPT}_{\rho, \mathcal{L}}(M).$$

Using $\text{OBJ}(T_\rho F_v) \leq \lambda_1$,

$$\mathbb{P}_{v \in \mathcal{S}}(\text{OBJ}(T_\rho F_v) > C_{\tau, p} + \varepsilon) > \varepsilon/\lambda_1. \quad (66)$$

If $v \in \mathcal{S}$ satisfies $\text{OBJ}(T_\rho F_v) > C_{\tau, p} + \varepsilon$, then by definition of $C_{\tau, p}$ in (57) and Lemma 4.3, there exists an $i_0 \in \{1, \dots, m\}$ such that $\text{Inf}_{i_0} T_\rho F_v > \tau$. Using convexity of the function $A \mapsto \text{Tr}(AA^*)$ for $A \in M_n(\mathbb{C})$, we see from the definition (7) of the influence that $\text{Inf}_{i_0}(f)$ is a convex function of f . Therefore,

$$\tau < \text{Inf}_{i_0} T_\rho F_v = \text{Inf}_{i_0}(\mathbb{E}_{(v, w) \in \mathcal{E}} T_\rho F_w \circ \pi_{vw}) \leq \mathbb{E}_{(v, w) \in \mathcal{E}}(\text{Inf}_{i_0} T_\rho F_w \circ \pi_{vw}).$$

Using $\text{Inf}_{i_0} T_\rho F_w \circ \pi_{vw} \leq n$, we deduce that if $v \in \mathcal{S}$ satisfies $\text{OBJ}(T_\rho F_v) > C_{\tau, p} + \varepsilon$ then for this fixed v ,

$$\mathbb{P}_{(v, w) \in \mathcal{E}}(\text{Inf}_{i_0} T_\rho F_w \circ \pi_{vw} > \tau/2) > \tau/(2n). \quad (67)$$

For any $w \in \mathcal{W}$, define a set of labels $L(w) \subseteq \{1, \dots, m\}$ by

$$L(w) := \{i \in \{1, \dots, m\} : \text{Inf}_i T_\rho F_w > \tau/2\},$$

and note that if v, w and i are such that $\text{Inf}_i T_\rho F_w \circ \pi_{vw} > \tau/2$ then $\text{Inf}_{\pi_{vw}^{-1}i} T_\rho F_v > \tau/2$, i.e. $\pi_{vw}^{-1}i \in L(w)$. We now define a labeling $\eta: \mathcal{S} \cup \mathcal{W} \rightarrow \{1, \dots, m\}$. If $v \in \mathcal{S}$ satisfies $\text{OBJ}(T_\rho F_v) > C_{\tau, p} + \varepsilon$, then as shown above there exists an $i_0 \in \{1, \dots, m\}$ such that $\text{Inf}_{i_0} T_\rho F_v > \tau$, and we let $\eta(v) := i_0$; otherwise we define $\eta(v)$ arbitrarily. For $w \in \mathcal{W}$, let $\eta(w)$ be chosen uniformly at random in $L(w)$ in case $L(w) \neq \emptyset$, and arbitrarily otherwise.

Since $\|F_w(\sigma)\| \leq 1$ for all $\sigma \in \{-1, 1\}^m$ we have $\|F_w\|_{L_2, \mathcal{G}} \leq n$ for all $w \in \mathcal{W}$. Therefore, as noted after (11),

$$\sum_{i=1}^m \text{Inf}_i T_\rho F_w = \sum_{S \subseteq \{1, \dots, m\}} |S| \rho^{|S|} \text{Tr}(|\widehat{F}_w(S)|^2) \leq n(\max_{t \geq 0} t \rho^t) \leq n(1 - \rho)^{-1},$$

and $|L(w)| \leq 2\tau^{-1}n(1 - \rho)^{-1}$. By definition of η any $(v, w) \in \mathcal{E}$ such that both events in (66) and (67) hold leads to a satisfied edge with probability at least $|L(w)|^{-1}$, hence we have shown that the unique games instance has value at least $(\varepsilon\tau/(2n))\lambda_1^{-1}(\tau/2)(1 - \rho)n^{-1}$. \square

Proof of Theorem 1.14. Let $K = K_{\mathbb{C}} = 2$ be the infimum of all constants $K_{\mathbb{C}}$ such that (4) holds for all PSD M . By definition of K and λ_1 there exists an n and a PSD matrix $M \in M_{n \times n}(\mathbb{C})$ such that $\lambda_1/\text{OPT}(M) > K - \delta/2$.

For an instance \mathcal{L} of Unique Games and appropriately chosen ρ and c, s , consider the problem of deciding whether $\text{OPT}_{\rho, \mathcal{L}}(M) > s$ or $\text{OPT}_{\rho, \mathcal{L}}(M) < c$, where $\text{OPT}_{\rho, \mathcal{L}}$ is defined in (65).

If \mathcal{L} has an $(1 - \alpha)$ -satisfying labeling for some $0 < \alpha < 1$, then by Lemma 5.2 it holds that

$$\text{OPT}_{\rho, \mathcal{L}}(M) \geq (1 - \alpha)\rho\lambda_1. \quad (68)$$

Conversely, assume that no assignment satisfies more than a fraction β of edges in \mathcal{L} . By the contrapositive of Lemma 5.3, as long as

$$\varepsilon\tau^2(1 - \rho)n^{-2}\lambda_1^{-1}/4 > \beta \quad (69)$$

²The supremum in (65) is always achieved.

it must be that $\text{OPT}_{\rho, \mathcal{L}}(M) \leq C_{\tau, p} + 2\varepsilon$. Choosing τ small enough (depending on ε and n) so that Lemma 4.4 applies, we deduce that

$$\text{OPT}_{\rho, \mathcal{L}}(M) \leq (1 + \varepsilon)\text{OPT}(M) + 2\varepsilon. \quad (70)$$

In summary, deciding whether $\text{OPT}_{\rho, \mathcal{L}}(M) > (1 - \alpha)\rho\lambda_1$ or $\text{OPT}_{\rho, \mathcal{L}}(M) < (1 + \varepsilon)\text{OPT}(M) + 2\varepsilon$ could disprove the Unique Games Conjecture (UGC). That is, approximating the quantity $\text{OPT}(M)$ within a multiplicative factor smaller than $(1 - \alpha)\rho\lambda_1 / ((1 + \varepsilon)\text{OPT}(M) + 2\varepsilon)$ is as computationally hard as disproving UGC.

Now we choose parameters. First select ε small enough so that $(1 + \varepsilon)\text{OPT}(M) + 2\varepsilon \leq \text{OPT}(M)(1 + \delta/8)$, and then select α small enough and ρ close enough to 1 so that $(1 - \alpha)\rho(K - \delta/2) > K - 3\delta/4$. Finally, choose τ , and p , so that the application of Lemma 4.4 in (70) is justified, and β small enough so that (69) holds. In summary for any $\delta > 0$ assuming UGC is it is always possible to find a family of instances \mathcal{L} with parameters α and β , hence a family of instances of NCGI with parameters ρ and c, s such that all constraints above are satisfied, i.e. $c/s > K - \delta$ and it is NP-hard to decide whether $\text{OPT}_{\rho, \mathcal{L}}(M) > s$ or $\text{OPT}_{\rho, \mathcal{L}}(M) < c$ for any $\delta > 0$. \square

Proof of Theorem 1.12. Theorem 1.12 follows immediately from Theorem 1.14, since [BRS15, Theorem 1.2] implies that $K = K_{\mathbb{C}} = 2$ in Theorem 1.14. \square

Proof of Theorem 1.16. For each $i \in \{1, \dots, n\}$, let $f_i, h_i: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$ and let $U_i \in M_n(\mathbb{C}^N)$. Theorem 1.16 is proven in the same way as Theorem 1.14, using slightly different definitions for the main quantities used in the proof. Specifically, for any (f_1, \dots, f_n) , define

$$\mathfrak{B}_{ij} \left(\sum_{S \subseteq \{1, \dots, m\}} \iota(\widehat{f_i}(S)) W_S \right) := \sum_{S \subseteq \{1, \dots, m\}: |S|=1} \overline{\iota(\widehat{f_i}(S))} \iota(M_{ij}) W_S, \quad (71)$$

and for $0 < \rho < 1$ and $p > n$ define

$$\text{OBJ}((f_1, \dots, f_n), (h_1, \dots, h_n)) := \sum_{i, j=1}^n 2^{-m} \sum_{\sigma \in \{-1, 1\}^m} \text{Tr}[T_{\rho} f_i^{\iota}(\sigma) \mathfrak{B}_{ij} T_{\rho} h_j^{\iota}(\sigma)]. \quad (72)$$

Finally, define

$$\begin{aligned} C_{\tau, p}(M) := \sup \left\{ \mathbb{E} \sum_{i, j}^n \text{Tr}(\iota(M_{ij})) \left[\text{Chop} T_{\rho} Q_{f_i}^{\iota}(\langle g_1, U_i \otimes I \rangle H_1, \dots, \langle g_m, U_i \otimes I \rangle H_m) \right] \right. \\ \left. \cdot \left[\text{Chop} T_{\rho} Q_{f_j}^{\iota}(\langle g_1, U_i \otimes I \rangle H_1, \dots, \langle g_m, U_i \otimes I \rangle H_m) \right]^* \right\} : \\ f_i: \{-1, 1\}^m \rightarrow \mathcal{U}_n(\mathbb{C}), \quad \max_{i, j=1, \dots, m} \text{Inf}_i f_j \leq \tau \}. \end{aligned} \quad (73)$$

\square

6. MAXIMIZING NONCOMMUTATIVE NOISE STABILITY

By adapting the proof of the Majority is Stablest Theorem from [MOO10], we can get the following consequence of Corollary 3.8.

Corollary 6.1. *Let $0 \leq \rho < 1$ and let $\varepsilon > 0$. Let $\delta = 20n^{1/2}\tau^{\frac{1-\rho}{30(c_2c_3)}} + O_{m,n}(\tau^{-1/3}p^{-1/2})$. Then there exists $\tau > 0$ such that, if $f: \{-1, 1\}^m \rightarrow M_n(\mathbb{C})$ satisfies $\|f(\sigma)\| \leq 1$ for all $\sigma \in \{-1, 1\}^m$, $\mathbb{E}_{b_i \sim \mathcal{B}} f\{b_i\} = 0$, and $\max_{i=1, \dots, m} \text{Inf}_i(f) < \tau$, then*

$$\frac{1}{n} \mathbb{E}_{b_i \sim \mathcal{B}} \text{Tr} |T_\rho Q_f\{b_i\}|^2 \leq \frac{1}{n} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} |\text{Chop} T_\rho Q_f^t\{G_i H_i\}|^2 + O(\varepsilon + \delta). \quad (74)$$

Moreover, $|\frac{1}{n} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} \text{Chop} T_\rho Q_f^t\{G_i H_i\}| \leq \delta$.

Remark 6.2. In the case $n = p = 1$, it is known from [MOO10] that the right-hand side of (74) is $\frac{2}{\pi} \arcsin \rho + \varepsilon$. For larger n , the left side of (74) can be interpreted as the noise stability of Q_f with discrete inputs, and the right side as the noise stability of a function with operator norm pointwise bounded by 1 under random Gaussian matrix inputs. Eq. (74) can thus be thought of as a matrix-valued version of one of the two main steps in the proof of the Majority is Stablest Theorem. However, for larger n , there seems to be no version of Borell's isoperimetric inequality that describes what the right-hand side of (74) should be. (Recall that Borell's isoperimetric inequality states that the noise stability of a subset of Euclidean space of fixed Gaussian measure is maximized when the set is a half space.)

Proof. Since $\mathbb{E}_{b_i \sim \mathcal{B}} f\{b_i\} = 0$, we have $\mathbb{E}_{b_i \sim \mathcal{B}} Q_f\{b_i\} = 0$. Using the Cauchy-Schwarz inequality and Corollary 3.8,

$$\begin{aligned} & \left| \frac{1}{n} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} |T_\rho Q_f\{G_i H_i\}|^2 - \frac{1}{n} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} |\text{Chop} T_\rho Q_f^t\{G_i H_i\}|^2 \right| \\ &= \left| \frac{1}{n} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} |T_\rho Q_f^t\{G_i H_i\}|^2 - \frac{1}{n} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} \left(\text{Chop} T_\rho Q_f^t\{G_i H_i\} [T_\rho Q_f^t\{G_i H_i\}]^* \right) \right| \\ & \quad + \left| \frac{1}{n} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} \left(\text{Chop} T_\rho Q_f^t\{G_i H_i\} [T_\rho Q_f^t\{G_i H_i\}]^* \right) - \frac{1}{n} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} |\text{Chop} T_\rho Q_f^t\{G_i H_i\}|^2 \right| \\ &\leq \left(\|T_\rho Q_f^t\{G_i H_i\}\|_{2,\nu} + \|\text{Chop} T_\rho Q_f^t\{G_i H_i\}\|_{2,\nu} \right) \cdot \|T_\rho Q_f^t\{G_i H_i\} - \text{Chop} T_\rho Q_f^t\{G_i H_i\}\|_{2,\nu} \\ &\leq 20n^{1/2}\tau^{\frac{1-\rho}{30(c_2c_3)}} + O_{m,n}(\tau^{-1/3}p^{-1/2}). \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{1}{n} \mathbb{E}_{b_i \sim \mathcal{B}} \text{Tr} |T_\rho Q_f\{b_i\}|^2 \stackrel{(17) \wedge (20)}{=} \frac{1}{n} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} |T_\rho Q_f^t\{G_i H_i\}|^2 \\ & \leq \frac{1}{n} \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} |\text{Chop} T_\rho Q_f^t\{G_i H_i\}|^2 + 20n^{1/2}\tau^{\frac{1-\rho}{30(c_2c_3)}} + O_{m,n}(\tau^{-1/3}p^{-1/2}), \end{aligned}$$

proving (74).

Using $\mathbb{E}_{b_i \sim \mathcal{B}} Q_f^t \{b_i\} = \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} T_\rho Q_f^t \{G_i H_i\} = 0$ and the Cauchy-Schwarz inequality,

$$\begin{aligned} \frac{1}{n} \left| \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr Chop} T_\rho Q_f^t \{G_i H_i\} \right| &= \frac{1}{n} \left| \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr Chop} T_\rho Q_f^t \{G_i H_i\} - \mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} T_\rho Q_f^t \{G_i H_i\} \right| \\ &\leq 20n^{1/2} \tau^{\frac{1-\rho}{30(c_2 c_3)}} + O_{m,n}(\tau^{-1/3} p^{-1/2}), \end{aligned}$$

using Corollary 3.8 again. \square

7. AN ANTI-CONCENTRATION INEQUALITY

As in [MOO10], we can use our invariance principle to prove anti-concentration estimates of polynomials.

Corollary 7.1 (An Anti-Concentration Estimate). *There exists a constant $C > 0$ such that the following holds. Let $Q: (M_n(\mathbb{C}))^m \rightarrow M_n(\mathbb{C})$ be a noncommutative multilinear polynomial of degree d . Assume $\mathbb{E}_{b_i \sim \mathcal{B}} \frac{1}{n} \text{Tr} |Q\{b_i\}|^2 \leq 1$. Let $\tau = \max_{1 \leq j \leq m} \text{Inf}_j(Q)$. Define $\text{Var}(Q) = \mathbb{E}_{b_i \sim \mathcal{B}} \text{Tr} |Q\{b_i\} - (\mathbb{E}_{b_j \sim \mathcal{B}} Q\{b_j\})|^2$. Then, for any $t \in \mathbb{R}$,*

$$\begin{aligned} \frac{1}{n} \mathbb{P}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}}} (\|Q^t \{G_i H_i\}\| > t) &\leq \mathbb{P}_{b_i \sim \mathcal{B}} (\|Q\{b_i\}\| > t) + O(n^3 c_3^d \tau^{1/100}) \\ &\quad + Cd(4\tau^{1/100} n / [\text{Var}(Q)]^{1/2})^{1/d} + O_{m,n}(\tau^{-1/100} p^{-1/2}). \end{aligned}$$

Remark 7.2. The $\frac{1}{n}$ term on the left side of the inequality seems to be an artifact of our proof method. It comes from (77) below, where we bound the normalized trace of a matrix by its operator norm.

Proof. Define $\phi: \mathbb{R} \rightarrow \mathbb{R}$ by $\phi(x) = c \cdot \exp\left(\frac{-1}{1-x^2}\right)$ if $|x| < 1$ and $\phi(x) = 0$ for all other $x \in \mathbb{R}$. The constant $1/2 < c < 4$ is chosen so that $\int_{\mathbb{R}} \phi(x) dx = 1$. It is well-known that ϕ is an infinitely differentiable function with bounded derivatives.

Fix $r, s > 0$. Define $\psi: \mathbb{R} \rightarrow \mathbb{R}$ by

$$\psi(x) = \begin{cases} 0 & \text{if } x \leq r - s \\ \frac{s-r+x}{2s} & \text{if } r - s \leq x \leq r + s \\ 1 & \text{if } x > r + s. \end{cases} \quad (75)$$

Define $\psi_\lambda(x) = \psi * \phi_\lambda(x) = \int_{\mathbb{R}} \psi(y) \phi_\lambda(x - y) dy$. Then $\psi_\lambda(x) = \psi(x)$ for any $x \in \mathbb{R}$ with $x > r + s + \lambda$ or $x < r - s - \lambda$. So,

$$\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \text{Tr} \psi_\lambda |Q^t \{G_i H_i\}|^2 \geq \mathbb{P}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}}} (\|Q^t \{G_i H_i\}\| > r + s + \lambda). \quad (76)$$

$$\mathbb{E}_{b_i \sim \mathcal{B}} \frac{1}{n} \text{Tr} \psi_\lambda |Q\{b_i\}|^2 \leq \mathbb{P}_{b_i \sim \mathcal{B}} (\|Q\{b_i\}\| > r - s - \lambda). \quad (77)$$

Note that $|\psi_\lambda'''(x)| \leq 10^{10} \lambda^{-2} s$. Applying Theorem 3.5,

$$\begin{aligned} &\mathbb{E}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}_p}} \frac{1}{n} \text{Tr} \psi_\lambda |Q^t \{G_i H_i\}|^2 \\ &\leq \mathbb{E}_{b_i \sim \mathcal{B}} \frac{1}{n} \text{Tr} \psi_\lambda |Q\{b_i\}|^2 + s \lambda^{-2} n^{3/2} 10^{10} (5^6 c_3)^d \tau^{1/4} + O_{m,n}(\lambda^{-2} s p^{-1/2}). \end{aligned} \quad (78)$$

Combining (76), (77) and (78),

$$\begin{aligned}
& \frac{1}{n} \mathbb{P}_{\substack{G_i \sim \mathcal{G} \otimes I \\ H_i \sim \mathcal{H}}} (\|Q'\{G_i H_i\}\| > r + s + \lambda) \\
& \leq \mathbb{P}_{b_i \sim \mathcal{B}} (\|Q'\{b_i\}\| > r - s - \lambda) + s\lambda^{-2} 10^{10} (5^6 c_3)^d \tau^{1/4} + O_{m,n}(s\lambda^{-2} p^{-1/2}) \\
& = \mathbb{P}_{b_i \sim \mathcal{B}} (\|Q\{b_i\}\| > r + s + \lambda) + \mathbb{P}_{b_i \sim \mathcal{B}} (r + s + \lambda > \|Q\{b_i\}\| > r - s - \lambda) \\
& \quad + s\lambda^{-2} n^{3/2} 10^{10} (5^6 c_3)^d \tau^{1/4} + O_{m,n}(s\lambda^{-2} p^{-1/2}). \tag{79}
\end{aligned}$$

It remains to show that the second term in (79) is small. To this end we apply the anti-concentration result of [CW01, Theorem 8] (with $q = 2d$ in their notation) to get: there exists an absolute constant $C' > 0$ such that, if g_1, \dots, g_m are i.i.d. standard real Gaussian random variables, and if Q is any noncommutative multilinear polynomial, then for all $\varepsilon > 0$,

$$\mathbb{P}_{g_1, \dots, g_m} (\|Q\{g_i\}\| < \varepsilon) \leq C' d(\varepsilon / [\mathbb{E}_{g_1, \dots, g_m} \|Q\{g_i\}\|^2]^{1/2})^{1/d}.$$

Since $\mathbb{E}_{g_1, \dots, g_m} \|Q\{g_i\}\|^2 \geq \mathbb{E}_{g_1, \dots, g_m} \frac{1}{n} \text{Tr} |Q\{g_i\}|^2 \geq \mathbb{E}_{g_1, \dots, g_m} \frac{1}{n} \text{Tr} |Q\{g_i\} - \widehat{Q}(\emptyset)|^2$, we conclude that, for any $r \in \mathbb{R}$, we have the following ‘‘small ball’’ probability estimate.

$$\mathbb{P}_{g_1, \dots, g_m} (|\|Q\{g_i\}\| - r| < \varepsilon) \leq C' d(\varepsilon n / [\mathbb{E}_{g_1, \dots, g_m} \text{Tr} |Q\{g_i\} - \widehat{Q}(\emptyset)|^2]^{1/2})^{1/d}. \tag{80}$$

Now, applying the invariance principle [IM12, Theorem 3.6] with the function $\Psi: M_n(\mathbb{C}) \rightarrow \mathbb{R}$ defined by

$$\Psi(A) = \begin{cases} 0 & \text{if } \|A\| \leq r - 2s \\ \frac{2s - r + \|A\|}{s} & \text{if } r - 2s \leq \|A\| \leq r - s \\ 1 & \text{if } r - s \leq \|A\| \leq r + s \\ \frac{2s + r - \|A\|}{s} & \text{if } r + s \leq \|A\| \leq r + 2s \\ 0 & \text{if } \|A\| > r + 2s. \end{cases}$$

We get

$$|\mathbb{E}_{b_i \sim \mathcal{B}} \Psi(Q\{b_i\}) - \mathbb{E}_{g_1, \dots, g_m} \Psi(Q\{g_i\})| \leq \frac{2}{s} n^3 C'' \tau^{1/50}. \tag{81}$$

So, applying the definition of Ψ to (81), we get

$$\begin{aligned}
\mathbb{P}_{b_i \sim \mathcal{B}} (|\|Q\{b_i\}\| - r| < s) & \leq \frac{2}{s} C'' n^3 \tau^{1/50} + \mathbb{P}_{g_1, \dots, g_m} (|\|Q\{g_i\}\| - r| < 2s) \\
& \stackrel{(80)}{\leq} \frac{2}{s} C'' n^3 \tau^{1/50} + C' d(2sn / [\mathbb{E}_{g_1, \dots, g_m} \text{Tr} |Q\{g_i\} - \widehat{Q}(\emptyset)|^2]^{1/2})^{1/d} \\
& = \frac{2}{s} C'' n^3 \tau^{1/50} + C' d(2sn / [\mathbb{E}_{b_i \sim \mathcal{B}} \text{Tr} |Q\{b_i\} - \widehat{Q}(\emptyset)|^2]^{1/2})^{1/d}.
\end{aligned}$$

Finally, substitute the last inequality into (79) and set $s = \lambda = \tau^{1/100}$. \square

Remark 7.3. The theorem [IM12, Theorem 3.6] used in (81) provides an extra multiplicative factor of 2^{n^2} in (81). However, this constant can be removed in the following way. Using their notation, they define a function $\phi: \mathbb{R}^k \rightarrow \mathbb{R}$ so that $\phi(x) = \exp(\frac{-1}{1-\|x\|_2^2})$ if $\|x\|_2 < 1$ and $\phi(x) = 0$ otherwise. (In the present paper, we set $k = n^2$.) This is the function they use in their convolution formula. If we instead use a function ϕ which is a product of

one-dimensional functions, each of which is supported in the interval $[-1, 1]$, e.g. $\phi(x) = \prod_{i=1}^k e^{\frac{-1}{1-x^2}}$, then the factor 2^k no longer appears in their proof.

Remark 7.4. The stronger, though more restrictive anti-concentration inequality

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}_{g_1, \dots, g_m} (\|Q\{g_i\}\| > t) - \mathbb{P}_{b_i \sim \mathcal{B}} (\|Q\{b_i\}\| > t) \right| \leq O(n^3 c_3^d \tau^{1/100}) + Cd(2\tau^{1/100} n / [\text{Var}(Q)]^{1/2})^{1/d}, \quad (82)$$

follows more directly from [CW01, Theorem 8] and [IM12, Theorem 3.6] by repeating the argument above. For example, if we use ψ defined in (75), then [IM12, Theorem 3.6] implies that

$$|\mathbb{E}_{b_i \sim \mathcal{B}} \psi(Q\{b_i\}) - \mathbb{E}_{g_1, \dots, g_m} \psi(Q\{g_i\})| \leq \frac{2}{s} n^3 C'' \tau^{1/50}.$$

Applying the definition of ψ to this inequality, we get

$$\left| \mathbb{P}_{g_1, \dots, g_m} (\|Q\{g_i\}\| > r + s + \lambda) - \mathbb{P}_{b_i \sim \mathcal{B}} (\|Q\{b_i\}\| > r - s - \lambda) \right| \leq \frac{2}{s} n^3 C'' \tau^{1/50}. \quad (83)$$

Therefore,

$$\begin{aligned} & \left| \mathbb{P}_{g_1, \dots, g_m} (\|Q\{g_i\}\| > r + s + \lambda) - \mathbb{P}_{b_i \sim \mathcal{B}} (\|Q\{b_i\}\| > r + s + \lambda) \right| \\ &= \left| - \mathbb{P}_{g_1, \dots, g_m} (r + s + \lambda > \|Q\{g_i\}\| > r - s - \lambda) \right. \\ & \quad \left. + \mathbb{P}_{g_1, \dots, g_m} (\|Q\{g_i\}\| > r + s + \lambda) - \mathbb{P}_{b_i \sim \mathcal{B}} (\|Q\{b_i\}\| > r - s - \lambda) \right| \\ &\leq \left| \mathbb{P}_{g_1, \dots, g_m} (r + s + \lambda > \|Q\{g_i\}\| > r - s - \lambda) \right| \\ & \quad + \left| \mathbb{P}_{g_1, \dots, g_m} (\|Q\{g_i\}\| > r + s + \lambda) - \mathbb{P}_{b_i \sim \mathcal{B}} (\|Q\{b_i\}\| > r - s - \lambda) \right|. \end{aligned}$$

The second term is bounded by (83) and the first term is bounded by (80), setting $s = \lambda = \tau^{1/100}$.

Remark 7.5. It would be desirable to upgrade Corollary 7.1 and (82) to the stronger inequalities presented in [MNV16]. We leave this research direction to future investigations.

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DEPARTMENT OF MATHEMATICS, UCLA, LOS ANGELES, CA 90095-1555
E-mail address: `heilman@math.ucla.edu`

DEPARTMENT OF COMPUTING AND MATHEMATICAL SCIENCES, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CA 91125-2100
E-mail address: `vidick@cms.caltech.edu`