MAXIMAL FUNCTION ESTIMATES OF NAOR AND TAO

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Abstract. We review some high dimensional maximal function estimates from [NT].

1. Introduction

We discuss high dimensional maximal function estimates. We begin with a few examples. Let $B=[-1,1]^n\subset\mathbb{R}^n$ be the cube. For $1\leq p\leq\infty$, consider the maximal function on $L_p(\mathbb{R}^n)$ associated to the cube:

$$Mf(x):=\sup_{r>0}\frac{1}{|rB|}\int_{rB+x}|f(y)|\,dy.$$

It is known that M maps $L_p(\mathbb{R}^n)$ boundedly to $L_p(\mathbb{R}^n)$, and that $||M||_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)}\leq c_p$

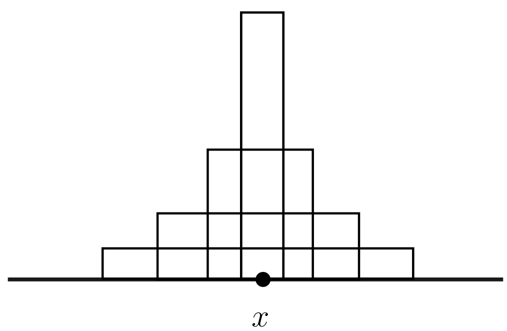

Figure 1. Functions $\frac{1}{|rB|}1_{rB+x}$, used in the definition of $Mf(x)$

for $3/2<p\leq\infty$. However, the weak type $(1,1)$ norm of M satisfies $||M||_{L_1(\mathbb{R}^n)\to L_{1,\infty}(\mathbb{R}^n)}\geq c(\log n)^{1-1/100}$. Now, let $1\leq q<\infty$ and let $B\subset\mathbb{R}^n$ be the unit ball in the ℓ_q^n norm, i.e. $B=\{x\in\mathbb{R}^n\colon\ ||x||_q\leq 1\}$. Then $||M||_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)}\leq c_p$ for $1<p\leq\infty$. The property that differentiates the cube from the other ℓ_q balls is volume-normalized surface area. Suppose $\widetilde{B}:=rB$ with r defined so that $\mathrm{vol}(\widetilde{B})=1$. Then the surface area of $\tilde{B}$ is of order $\sqrt{n}$. However, the surface area of the unit cube is of order n, and this large surface area defeats the method used for the other ℓ_q balls. Note that the ℓ_1 ball is not uniformly convex, so uniform convexity does not appear to play a role in these estimates. Also, it is unclear if the constant $3/2$ that appears above gives the limit of what can be proven, or if it is merely an artifact of the method that is used. To get the constant $3/2$, one proves dimension independent estimates for the dyadic maximal function for all $1<p$, and then transfers the result to the full maximal function, at the cost of increasing p to $3/2<p$ [B].

Date: March 20, 2012.

Since M is an averaging operator, we see that $||M||_{L_\infty \to L_\infty} \leq 1$. Suppose we had an inequality of the form $|\{x\colon Mf(x) > t\}| \leq (c/t)\, ||f||_1$. Then the Marcinkiewicz interpolation theorem would immediately imply $||M||_{L_p \to L_p} \leq c \cdot c_p$, where c_p does not depend on n. So, to get an infinite family of L_p inequalities that are independent of n, it suffices to prove a weak type $(1,1)$ inequality that is independent of n. It is for this reason that we look for dimension independent weak type $(1,1)$ inequalities. However, note that this task appears quite difficult.

Remark 1.1. To apply the interpolation theorem, it actually suffices to prove that M is of restricted weak type $(1,1)$. Though this task may at first seem easier to accomplish, in practice it does not seem any easier than simply proving the weak type estimate.

There are only a few nontrivial dimension independent weak type $(1,1)$ inequalities that exist. For example, if T_t is a heat diffusion semigroup (similar to a T_t defined in class), then we have the Dunford-Hopf-Schwarz abstract maximal inequality

$$\left|\left\{x\colon \sup_{N>0} \frac{1}{N}\int_0^N T_t f(x) dt > \lambda\right\}\right| \leq \frac{1}{\lambda}\, ||f||_1 .$$

As a consequence of this, one can prove a dimension independent maximal inequality for the Poisson semigroup, and also dimension independent Littlewood Paley estimates. For $f\colon \mathbb{R}^n \to \mathbb{R}$ define $P_1(x) := c_n/(1+||x||_2^2)^{(n+1)/2}$, where c_n is defined so that $\int P_1 = 1$. Let $P_t(x) = t^{-n} P_1(x/t)$, and let $P_t f(x) := f * P_t(x)$. Then [S2, S1]

$$||\sup_{t>0} P_t f||_{1,\infty} \leq c\, ||f||_1 .$$

Moreover, for $s \geq 1, s \in \mathbb{Z}, L > 0$,

$$\left|\left|\left(\sum_j |P_{L2^{j+s}} * f - P_{L2^{j-s}} * f|^2\right)^{\frac{1}{2}}\right|\right|_p \leq c_{p,s}\, ||f||_p .$$

We now begin to discuss the weak type $(1,1)$ inequalities for the maximal function in general metric spaces. The approach of [NT] uses a combination of harmonic analysis and martingale techniques, similar to other methods that we have used in the course for completely different inequalities. The combination of harmonic analysis and martingales is not new [S1]. However, the addition of random partitions, the exact implementation of the martingales, and the generality of this result are quite novel.

We would like to generalize some of the above results to general metric measure spaces. However, before we even begin, we need to define what it means for a metric space to have dimension n. Such conditions should express some compatibility between the metric and the measure. In the end, the goal is to find geometric conditions, that only involve the metric and the measure, that allow one to prove these estimates. In doing so, we expect to understand the geometry of general metric spaces better.

We begin with some notation and definitions

Notation 1.2. For $A, B > 0$, $A \lesssim B$ means that there exists $C > 0$ such that $A \lesssim C \cdot B$. Also, $A \asymp B$ means $A \lesssim B$ and $B \lesssim A$.

Definition 1.3. A **metric measure space** (X, d, μ) is a separable metric space (X, d) together with a Radon measure μ. Let $B(x, r) := \{y \in X : d(x, y) \leq r\}$, and assume that $0 < \mu(B(x, r)) < \infty$ for all $r > 0$.

Definition 1.4. Let $p \geq 1$, and let $f : X \to \mathbb{C}$. The **weak L_p norm** of f is given by

$$||f||_{p,\infty} := \sup_{t>0} t \cdot [\mu(x\colon\ |f(x)| > t)]^{1/p}.$$

Note that $t^p \mu(|f| > t) \leq \int |f|^p$, so $||f||_{p,\infty} \leq ||f||_p$. For an operator M and for $p \geq 1$, we define the **weak (p, p) operator norm** $||M||_{p \to p,\infty}$ of M as the smallest real number $0 \leq ||M||_{p\to p,\infty} \leq \infty$ such that the following inequality holds

$$||Mf||_{p,\infty} \leq ||M||_{p\to p,\infty} ||f||_p .$$

Definition 1.5. Let $f : X \to \mathbb{C}$ be locally integrable. The **Hardy-Littlewood maximal function** is defined as

$$Mf(x) := \sup_{r>0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f(y)| \, d\mu(y).$$

Let $R \subset (0, \infty)$ be a set. Define the maximal operator M_R by the formula

$$M_R f(x) := \sup_{r\in R} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f(y)| \, d\mu(y).$$

Below, we will especially consider $R = 2^{\mathbb{Z}} = \{2^j : j \in \mathbb{Z}\}$ and $R = (0, 1]$.

It turns out that a general metric space has no chance of satisfying a reasonable weak type $(1, 1)$ bound. We therefore need some assumptions on the space that are mild enough to cover several cases, but specific enough so that it is possible to prove something. Formulating such conditions actually remains a challenge, as we will see. The following conditions impose some natural compatibility between the metric and the measure.

Definition 1.6. The space (X, d, μ) is called **Ahlfors David n-regular** if: for all $x \in X$ and for all $r > 0$,

$$r^n \leq \mu(B(x, r)) \leq Cr^n.$$

That is, μ scales like renormalized Lebesgue measure in Euclidean space. Also, replacing μ by $c \cdot \mu$ does not change the weak $(1, 1)$ norm of an operator M, so for the purpose of proving such an inequality, we may similarly rescale a measure satisfying $\mu(B(x, r)) \geq (1/c)r^n$ so that the lower bound $\mu(B, x, r) \geq r^n$ holds.

Definition 1.7. The space (X, d, μ) is called **n-microdoubling with constant** K if: for all $x \in X$ and all $r > 0$,

$$\mu(B(x, (1 + 1/n)r)) \leq K \cdot \mu(B(x, r))$$

Note that an Ahlfors David n-regular space (with constant C) is n-microdoubling (with constant eC).

Definition 1.8. The space $(X, d\mu)$ is called **strong n-microdoubling with constant** K if: for all $x \in X$, for all $r > 0$, and for all $y \in B(x, r)$

$$\mu(B(y, (1 + 1/n)r)) \leq K \cdot \mu(B(x, r)).$$

Remark 1.9. Perhaps some control on the derivative in r of $\mu(B(x,r))$ would lead to stronger estimates. Such a derivative bound would serve as the analogue of a bound on volume normalized surface area, which we discussed above.

Remark 1.10. Notice also that the above conditions are all scale invariant. That is, given $c, c' > 0$ we may replace the metric d with $c \cdot d$, and we may replace the measure μ with $c' \cdot \mu$, and then Definitions 1.5, 1.7 and 1.8 are unchanged. Moreover, the weak (p,p) operator norm is unchanged by these scalings. It is an important feature of our definitions that they remain scale invariant.

From harmonic analysis class, we know that a Vitali covering argument gives the following bound in Euclidean space, which worsens as n increases

$$||M||_{L_1(\ell_2^n) \to L_{1,\infty}(\ell_2^n)} \leq 5^n.$$

This covering argument can be sharpened to give the bound $n \log n$ for any norm on $\mathbb{R}^n$ [SS]. However, one cannot improve these bounds by covering arguments alone. Something more is necessary. For example, note that the heat diffusion semigroup on $\mathbb{R}^n$ is given by $T_t f(x) = f * h_t := f * ((4\pi t)^{-n/2} e^{-||y||_2^2/4t})(x)$, and the level sets of h_t are Euclidean balls. So, one can compare the averages over Euclidean balls to the average $f * h_t$, and then apply the Dunford-Hopf-Schwarz maximal inequality with a loss of a factor of n, so that $||M||_{L_1(\ell_2^n) \to L_{1,\infty}(\ell_2^n)} \leq n$. However, even in the Euclidean case, this is the best known weak type bound.

Theorem 1.11. *(Theorem 1.1, Localization,* [NT]*) Let $n \geq 1$ and $K \geq 5$. Let (X, d, μ) be a metric measure space satisfying the n-microdoubling condition from Definition 1.7. Fix a nonempty bounded $R \subseteq (0, \infty)$, and let $p \geq 1$. Then*

$$||M_R||_{L_p(X) \to L_{p,\infty}(X)} \lesssim K + \left(1 + \frac{\log \log K}{1 + \log n}\right)^{1/p} \cdot \sup_{r>0} \left|\left|M_{R \cap [r, nr]}\right|\right|_{L_p(X) \to L_{p,\infty}(X)}. \qquad (1)$$

By splitting the interval $[r, nr]$ into m pieces of the form $[rn^{j/m}, rn^{(j+1)/m}]$, one can deduce the following.

Theorem 1.12. *(Corollary 1.2,* [NT]*) Let $n \geq 1$ and $K \geq 5$. Let (X, d, μ) be a metric measure space satisfying the strong n-microdoubling condition from Definition 1.8. Then*

$$||M||_{1 \to 1,\infty} \lesssim_K n \log n,$$

$$||M_{2^{\mathbb{Z}}}||_{1 \to 1,\infty} \lesssim_K \log n.$$

2. A Variant of Doob's Maximal Inequality

In this section, we begin with the following.

Assumption 1. (X, d, μ) is a metric measure space with $\mu(X) < \infty$.

Recall the classical Doob maximal inequality. This inequality is often proven using stopping times. We present a more analytic proof. The key point is that, since we are dealing with a sequence of partitions that refine each other, we are in the setting of an ultrametric space, so we get a constant of 1 in a weak-type inequality.

Theorem 2.1. ***(Doob Maximal inequality)*** *Let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \mathcal{F}_N$ be an increasing sequence of (finitely generated) σ-algebras (i.e. a sequence of finer and finer partitions). Let $f\colon X \to \mathbb{R}$, $f \geq 0$. Then*

$$||\sup_{k\geq 0} |\mathbb{E}(f|\mathcal{F}_k)|\,||_{1,\infty} \leq ||f||_1 .$$

Using this, and that $\left|\left|\sup_{k\geq 0} |\mathbb{E}(f|\mathcal{F}_k)|\right|\right|_\infty \leq ||f||_\infty$, we get, for $1 < p \leq \infty$,

$$||\sup_{k\geq 0} |\mathbb{E}(f|\mathcal{F}_k)|\,||_p \leq \frac{p}{p-1}\,||f||_p .$$

Proof. Let $t > 0$ and let $A := \{x\colon \sup_{k\geq 0} |\mathbb{E}(f|\mathcal{F}_k)(x)| > t\}$. For each $x \in X$, choose $k = k(x)$ so that

$$\left|\mathbb{E}(f|\mathcal{F}_{k(x)})(x)\right| > t. \qquad (*)$$

Let $P_j(x)$ denote the element of $\mathcal{F}_j$ such that $x \in P_j(x)$. Then $A \subseteq \cup_{x\in A} P_{k(x)}(x)$. Let $x, y \in A$. If $P_j(x) \cap P_\ell(y) \neq \emptyset$, then either $P_j(x) \subseteq P_\ell(y)$ or $P_\ell(y) \subseteq P_j(x)$, by the nesting property of the σ-algebras. So, there exists $A' \subset A$ such that the following union is disjoint: $\cup_{x\in A'} P_{k(x)}(x) \supseteq A$. From $(*)$, since $E(f|\mathcal{F}_{k(x)})(x) = (1/\mu(P_{k(x)}(x))) \int_{P_{k(x)}(x)} |f| > t$,

$$\mu(A) \leq \sum_{x\in A'} \mu(P_{k(x)}(x)) \leq \frac{1}{t} \sum_{x\in A'} \int_{P_{k(x)}(x)} |f| \leq \frac{1}{t}\,||f||_1 .$$

Here we have used disjointness of the $P_{k(x)}(x)$ for $x \in A'$. So, the weak type $(1,1)$ inequality is complete.

Now, let $g(x) = f(x)1_{|f(x)|>t/2}$, and denote $M(f) := \sup_{k\geq 0} |\mathbb{E}(f|\mathcal{F}_k)|$. Then $Mf \leq Mg + t/2$, so $\{x\colon Mf(x) > t\} \subset \{Mg(x) > t/2\}$, and from above,

$$\mu(x\colon Mf(x) > t) \leq \frac{2}{t} \int |f|\, 1_{\{|f|>t/2\}} d\mu.$$

Therefore, (achieving a slightly different constant than above)

$$\begin{aligned}
\int (Mf)^p &= \int_0^\infty pt^{p-1} \mu(Mf \geq t) dt \\
&\leq \int_0^\infty pt^{p-1} \frac{2}{t} \int_X f 1_{\{|f|>t/2\}} d\mu dt \quad \text{, Doob} \\
&= 2 \int_X f \int_0^{2|f|} pt^{p-2} dt d\mu \quad \text{, Fubini} \\
&= \frac{p}{p-1} 2^p \int_X f^p d\mu.
\end{aligned}$$

□

We now give a modification of Doob's inequality that we will need later on. In some sense, this is a stopping time argument, but we will not emphasize this below.

Theorem 2.2. ***(Modified Doob Inequality)*** [NT]. *Let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \mathcal{F}_N$ be an increasing sequence of (finitely generated) σ-algebras, and let $1 \leq p < \infty$. For each $k \in \mathbb{N}$ let M_k be a sublinear operator defined on $L_p(X) + L_\infty(X)$. (So, for all functions f, g in the domain of*

M_k, and for all $c \in \mathbb{R}$, the following holds pointwise: $|M_k(f+g)| \le |M_k(f)| + |M_k(g)|$ and $|M_k(cf)| = |c| \cdot |M_k f|$.) Assume that

$$f \in L_p(X) \Rightarrow ||M_k f||_{p,\infty} \le A\,||f||_p\,, \tag{2}$$

$$f \in L_\infty(X) \Rightarrow ||M_k f||_\infty \le B||\mathbb{E}(|f|\,|\mathcal{F}_k)||_\infty. \tag{3}$$

*Suppose also that the following **localization property** holds*

$$f \in L_p(X) + L_\infty(X) \text{ and } E_k \in \mathcal{F}_k \Rightarrow 1_{E_k} M_{k+1} f = M_{k+1}(1_{E_k} f). \tag{4}$$

Then, for all $f \in L_p(X)$,

$$||\sup_{k\ge 0} |M_k f|\,||_{p,\infty} \le ((2A)^p + (2B)^p)^{1/p}\,||f||_p\,.$$

Proof. By replacing a given f by f/t for $t > 0$, it suffices to show

$$f \in L_p(X) \Rightarrow \mu\left(\sup_{k\ge 0} |M_k f| > 1\right) \le ((2A)^p + (2B)^p)\int |f|^p\,. \tag{5}$$

Let $f \in L_p(X) \cap L_\infty(X)$. By Jensen's inequality, $[\mathbb{E}(|f|\,|\mathcal{F}_k)]^p \le \mathbb{E}(|f|^p\,|\mathcal{F}_k)$, so Doob (Thm. 2.1) says

$$\mu\left(\sup_{k\ge 0} \mathbb{E}(|f|\,|\mathcal{F}_k) \ge \frac{1}{2B}\right) \le \mu\left(\sup_{k\ge 0} \mathbb{E}(|f|^p\,|\mathcal{F}_k) \ge \frac{1}{(2B)^p}\right) \le (2B)^p \int |f|^p\,.$$

So, to get (5), it suffices to show

$$\mu(D) := \mu\left(\left\{\sup_{k\ge 0} |M_k f| > 1\right\} \setminus \left\{\sup_{k\ge 0} \mathbb{E}(|f|\,|\mathcal{F}_k) \ge \frac{1}{2B}\right\}\right) \le (2A)^p \int |f|^p\,. \tag{6}$$

Observe

$$D \subseteq \bigcup_{k=0}^{N} \left\{|M_k f| > 1 \text{ and } \sup_{0\le j<k} \mathbb{E}(|f|\,|\mathcal{F}_j) < \frac{1}{2B}\right\}. \tag{7}$$

Now, define

$$A_k := X \setminus \bigcup_{0\le j<k} \left\{\mathbb{E}(|f|\,|\mathcal{F}_j) \ge \frac{1}{2B}\right\}, \text{ and } \Omega_k := \left\{\mathbb{E}(|f|\,|\mathcal{F}_k) \ge \frac{1}{2B}\right\} \cap A_k.$$

Note that $A_k \in \mathcal{F}_{k-1}$, and the sets $\Omega_k \in \mathcal{F}_k$ are disjoint, since Ω_k occurs when the k^{th} conditional expectation is the first one to exceed $1/2B$. Now, the definitions of our sets, (7) and (4) show that

$$D \subseteq \bigcup_{k=0}^{N} \{|1_{A_k} M_k f| > 1\} = \bigcup_{k=0}^{N} \{|M_k(1_{A_k} f)| > 1\}. \tag{8}$$

Applying (3) and the definition of Ω_k, we have

$$||M_k(f 1_{A_k\setminus\Omega_k})||_\infty \le B||\mathbb{E}(|f|\,1_{A_k\setminus\Omega_k}|\mathcal{F}_k)||_\infty = B||\mathbb{E}(|f|\,|\mathcal{F}_k)1_{A_k\setminus\Omega_k}||_\infty. \le B/2B = 1/2$$

So, writing $f1_{A_k} = f1_{A_k\setminus\Omega_k} + f1_{\Omega_k}$, we have the following inclusion, less measure zero sets

$$\{|M_k(f1_{A_k})| > 1\} \subseteq \{|M_k(f1_{A_k\setminus\Omega_k}) > 1/2|\} \cup \{|M_k(f1_{\Omega_k})| > 1/2\} = \{|M_k(f1_{\Omega_k})| > 1/2\}. \tag{9}$$

We can finally prove (6) and thereby finish the proof. Applying (8), (9), (2), and disjointness of the Ω_k,

$$\mu(D) \leq \sum_{k=0}^{N} \mu\left(|M_k(f 1_{\Omega_k})| > 1/2\right) \leq \sum_{k=0}^{N} (2A)^p \int_{\Omega_k} |f|^p \leq (2A)^p \int |f|^p .$$

□

3. The Random Partition Lemma

During the last class, Professor Naor proved the following lemma (more or less). We only briefly discuss how this version of the lemma differs from the one presented in class.

Let (X, d, μ) be a metric measure space with bounded diameter. For a partition $\mathcal{P}$ of X and $x \in X$, we denote $\mathcal{P}(x)$ as the unique element of $\mathcal{P}$ containing x. We say that a sequence $\{\mathcal{P}_k\}_{k \geq 0}$ of partitions of X is a **partition tree** if the following conditions hold:

- $\mathcal{P}_0$ is the trivial partition $\{X\}$
- For every $x \in X$ and $k \in \mathbb{Z}_{\geq 0}$ we have $\operatorname{diam}(\mathcal{P}_k(x)) \leq \operatorname{diam}(X)/2^k$
- For every $k \in \mathbb{Z}_{\geq 0}$, the partition $\mathcal{P}_{k+1}$ is a refinement of the partition $\mathcal{P}_k$, i.e. for every $x \in X$ we have $\mathcal{P}_{k+1}(x) \subseteq \mathcal{P}_k(x)$

Definition 3.1. For $\beta > 0$, a probability distribution over partition trees $\{\mathcal{P}_k\}_{k \geq 0}$ is said to be **β-padded** if, for every $x \in X$ and every $k \in \mathbb{Z}_{\geq 0}$,

$$\mathbb{P}\left[B(x, \beta \operatorname{diam}(X)/2^k) \subseteq \mathcal{P}_k(x)\right] \geq 1/2.$$

Theorem 3.2. *(Lemma 3.1,* [NT]*) Let $n \geq 1, K \geq 5$. Let (X, d, μ) be a separable bounded-diameter metric measure space satisfying Definition 1.7. Then X admits a $1/(64n \log K)$-padded probability distribution over partition trees.*

The Theorem differs from the one in class as follows. By rescaling the metric on X, we may assume $\operatorname{diam}(X) = 1$. Since X is bounded, $\mu(X) < \infty$, and we rescale μ so that μ is a probability measure. Let $x_1, x_2, x_3, \ldots$ be points chosen uniformly and independently at random from X according to the measure μ. For each k, let r_k be a random variable uniformly distributed on $[2^{-k-2}, 2^{-k-1}]$. We assume that the r_k are independent. Let $\mathbb{P}$ denote the joint distribution of $(x_1, x_2, \ldots,), (r_1, r_2, \ldots)$.

For every $k \in \mathbb{Z}_{\geq 0}$ define a random variable $j_k \colon X \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by

$$j_k(x) := \inf\{j \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \colon d(x, x_j) \leq r_k\}.$$

For every $k \in \mathbb{Z}_{\geq 0}$ and $\ell_1, \ldots, \ell_k \in \mathbb{Z}_{\geq 0}$, define

$$P(\ell_1, \ldots, \ell_k) := \{x \in X \colon j_1(x) = \ell_1, \ldots, j_k(x) = \ell_k\}.$$

Then $\mathcal{P}_k := \{P(\ell_1, \ldots, \ell_k) \colon \ell_1, \ldots, \ell_k \in \mathbb{Z}_{\geq 0}\}$ is a partition of X satisfying the nesting property and diameter bound. That is, $P(\ell_1, \ldots, \ell_{k+1}) \subseteq P(\ell_1, \ldots, \ell_k)$, and

$$P(\ell_1, \ldots, \ell_k) \subseteq B(x_{\ell_k}, r_k) \subseteq B(x_{\ell_k}, 2^{-k-1}).$$

Let $\beta = 1/(64n \log K)$, and fix $\ell \in \{1, \ldots, k\}$. Arguing as in class, we eventually conclude

$$\mathbb{P}\left[\exists y \in B(x, \beta/2^k), j_\ell(x) \neq j_\ell(y)\right] \leq 1 - \left(\frac{\mu(B(x, 2^{-\ell-2} - \beta 2^{-k}))}{\mu(B(x, 2^{-\ell-1} + \beta 2^{-k}))}\right)^{\beta 2^{-(k-\ell)+3}} .$$

Since $\ell \leq k$ and $\beta \leq 1/25$, we get $2^{-\ell-1} + \beta 2^{-k} \leq (1+1/n)^{n+1}(2^{-\ell-2} - \beta 2^{-k})$. So, applying Definition 1.7 $(n+1)$ times, we get

$$\mu(B(x, 2^{-\ell-1} + \beta 2^{-k})) \leq K^{n+1}\mu(B(x, (2^{-\ell-2} - \beta 2^{-k}))).$$

Plugging this into the previous equation then allows us to continue the argument as presented in class.

4. The Weak Type Maximal Inequality

We can now begin the proof of the Main Theorem 1.11.

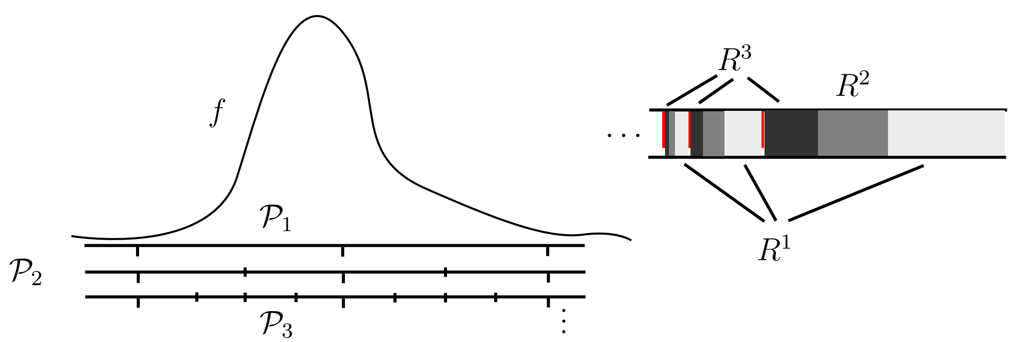

Figure 2. The partitions $\mathcal{P}_k$, and also the scales R^i.

Proof. Write $R \subseteq [0, D]$, $D > 1$, and fix $f \in L_p(X)$ with (distance)-bounded support. As before, by replacing f with f/t it suffices to prove

$$\mu(M_R f > 1) \lesssim C^p \left(\left(1 + \frac{\log\log K}{1 + \log n}\right) Q^p + K^p \right) \int |f|^p,$$

where $C > 0$ is a universal constant and

$$Q := \sup_{r>0} ||M_{R\cap[r,nr]}||_{L_p(X)\to L_{p,\infty}(X)}. \tag{10}$$

Step 1. Split M_R into thirds.

Let E be the support of f, and define

$$E' := \{x \in X : d(x, E) \leq D\}, \qquad E'' := \{x \in X : d(x, E) \leq 2D\}.$$

Then $E \subseteq E' \subseteq E''$ and $\operatorname{diam}(E'') \leq 4D + \operatorname{diam}(E) < \infty$, by our assumptions on D and f. By definition of R, $M_R f$ is supported in E', so $M_R f$ also has (distance)-bounded support. So, by enlarging the supports even more, the Theorem follows directly from the following

$$||M_R||_{L_p(E')\to L_{p,\infty}(E'')} \lesssim \left(1 + \frac{\log\log K}{1 + \log n}\right)^{1/p} Q + K.$$

By rescaling the metric, we may assume that $\operatorname{diam}(E'') = 1$. Given this rescaling, note that

$$M_{R\cap(1,\infty)} f(x) \leq \sup_{r>1} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| \leq \frac{1}{\mu(E'')} \int_{E''} |f(y)|\, d\mu(y).$$

So, by Jensen's inequality,

$$\left|\left|M_{R\cap(1,\infty)}f\right|\right|_{L_p(E'')}^p \leq \int_{E''} \frac{1}{\mu(E'')} \int_{E''} |f(y)|^p \, d\mu(y) d\mu(x) = ||f||_{L_p(E'')}^p .$$

That is, $\mu(M_{R\cap(1,\infty)}f > 1) \leq ||f||_p$, so

$$\mu(M_R f > 1) = \mu(\max\{M_{R\cap(0,1]}f, M_{R\cap(1,\infty)f}\} > 1) \leq \mu(M_{R\cap(0,1]}f > 1) + \mu(M_{R\cap(1,\infty)}f > 1).$$

And for the purpose of bounding $||M_R||_{L_p(E')\to L_{p,\infty}(E'')}$, we may assume that $R \subseteq (0,1]$.

We now apply Theorem 3.2. Let $\{\mathcal{P}_k\}_{k\geq 0}$ be a random partition tree on E'' that is β-padded with

$$\beta = \frac{1}{64n \log K}.$$

Let m be the largest integer such that $2^{-m} \leq \beta$. For $k \geq 0$ and $i \in \{1,2,3\}$, define

$$R_k^i := R \cap [2^{-(3k+i)m}, 2^{-(3k-1+i)m}] \quad \text{and} \quad R^i := \bigcup_{k\geq 0} R_k^i.$$

Then $R = R^1 \cup R^2 \cup R^3$, so

$$\begin{aligned}\mu(M_R f > 1) &= \mu(\max\{M_{R^1}f, M_{R^2}f, M_{R^3}f\} > 1) \\ &\leq \mu(M_{R^1}f > 1) + \mu(M_{R^2}f > 1) + \mu(M_{R^3}f > 1).\end{aligned} \tag{11}$$

We estimate each term of (11) separately. Fix $i \in \{1,2,3\}$ and $k \geq 0$, and define

$$E_k^i := \left\{x \in E' \colon M_{R_k^i}f(x) > 1\right\} \setminus \bigcup_{j=0}^{k-1} \left\{x \in E' \colon M_{R_j^i}f(x) > 1\right\}.$$

So, E_k^i is the first "time" that M_{R^i} exceeds 1. Thus, the sets E_k^i are disjoint and

$$\mu(M_{R^i}f > 1) = \mu\left(\sup_{k\geq 0} M_{R_k^i}f > 1\right) = \sum_{k\geq 0} \mu(E_k^i). \tag{12}$$

Step 2. Applying the partition estimates (and applying randomness).

We now replace the E_k^i with its randomization. Let

$$\widetilde{E}_k^i := \left\{x \in E_k^i \colon B\left(x, \frac{\beta}{2^{(3k+i+1)m}}\right) \subseteq \mathcal{P}_{(3k+i+1)m}(x)\right\}.$$

Observe, by Fubini and the definition of β-padded,

$$\mathbb{E}\mu(\widetilde{E}_k^i) = \iint 1_{\widetilde{E}_k^i} dP d\mu = \int_{E_k^i} \mathbb{P}\left[B\left(x, \frac{\beta}{2^{(3k+i+1)m}}\right) \subseteq \mathcal{P}_{(3k+i+1)m}(x)\right] d\mu(x) \geq \frac{\mu(E_k^i)}{2}. \tag{13}$$

For $g\colon E' \to \mathbb{R}$, define the sublinear operator

$$\widetilde{M}_{R_k^i} := 1_{\widetilde{E}_k^i} M_{R_k^i} 1_{E'}.$$

Combining (12) and (13), then using disjointness and definition of $\widetilde{E}_k^i$ and $\widetilde{M}_{R_k^i}$,

$$\mu(M_{R^i}f > 1) \leq 2\mathbb{E}\sum_{k\geq 0} \mu(\widetilde{E}_k^i) = 2\mathbb{E}\mu\left(\sup_{k\geq 0} \widetilde{M}_{R_k^i}f > 1\right). \tag{14}$$

Step 3. Randomized maximal functions, preparing for generalized Doob's inequality.

Let $r = 2^{-(3k+i)m}$ and let $v \asymp 1 + \frac{\log\log K}{1+\log n}$ be an integer such that $2^{m/v} \leq n$. (Recall $2^{-m} \asymp \beta = 1/(64 n \log K)$.) Let $g \in L_p(E')$, $t > 0$. By the definitions of Q, $\widetilde{M}_{R_k^i}$, and R_k^i,

$$\mu(\widetilde{M}_{R_k^i} g > t) \leq \mu(M_{R_k^i} g > t) = \mu(M_{R \cap [r, 2^m r]} g > t)$$
$$\leq \sum_{u=0}^{v-1} \mu(M_{R \cap [r 2^{\frac{um}{v}}, n r 2^{\frac{um}{v}}]} g > t) \leq v Q^p \left|\left|g\right|\right|_{L_p(E')}^p / t^p.$$

That is,

$$g \in L_p(E') \Rightarrow ||\widetilde{M}_{R_k^i} g||_{L_{p,\infty}(E')} \leq v^{1/p} Q \left|\left|g\right|\right|_{L_p(E')}. \tag{15}$$

Let $k \geq 0$ and define $\mathcal{F}_k := \sigma(\mathcal{P}_k)$ to be the σ-algebra generated by the partition $\mathcal{P}_k$. Then $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$. We claim: for every $k \geq 0$, if $F \in \mathcal{F}_{(3k+i+1)m}$ then

$$1_F \widetilde{M}_{R_{k+1}^i}(g) = \widetilde{M}_{R_{k+1}^i}(1_F g). \tag{16}$$

So, using the definition of $\widetilde{M}_{R_k^i}$, we claim: for almost every $x \in E'$,

$$1_F(x) 1_{\widetilde{E}_{k+1}^i}(x) M_{R_{k+1}^i}(g)(x) = 1_{\widetilde{E}_{k+1}^i}(x) M_{R_{k+1}^i}(1_F g)(x). \tag{17}$$

To check (17), we may let $x \in \widetilde{E}_{k+1}^i$, so that $B(x, \beta 2^{-(3k+i+1)m}) \subseteq \mathcal{P}_{(3k+i+1)m}(x)$. Now, since $F \in \mathcal{F}_{(3k+i+1)m}$ and $\mathcal{P}_{(3k+i+1)m}(x)$ is a basis element of the same σ-field, we know that $\mathcal{P}_{(3k+i+1)m}(x)$ is either disjoint from F or contained in F. (Note the use of the ultrametric condition here.) We consider each of the two cases separately. Recall that $2^{-m} \leq \beta$. If $\mathcal{P}_{(3k+i+1)m}(x) \subseteq F$, then for every $r \in R_{k+1}^i$, $r \leq 2^{-(3(k+1)-1+i)m}$, so

$$B(x,r) \subseteq B(x, 2^{-(3k+i+2)m}) \subseteq B(x, \beta 2^{-(3k+i+1)m}) \subseteq \mathcal{P}_{(3k+i+1)m}(x) \subseteq F. \tag{18}$$

The inclusion (18) shows in particular that $x \in F$, and also $M_{R_{k+1}^i}(1_F g) = M_{R_{k+1}^i}(g)$, so (17) holds. Now, in the case that $\mathcal{P}_{(3k+i+1)m}(x) \cap F = \emptyset$, then the containment $B(x,r) \subseteq \mathcal{P}_{(3k+i+1)m}(x)$ for all $r \in R_{k+1}^i$ shows that $B(x,r) \cap F = \emptyset$ for all $r \in R_{k+1}^i$. In particular, $x \notin F$, and $M_{R_{k+1}^i}(1_F g) = 0$, i.e. both sides of (17) are zero. In conclusion, (16) holds. So, in some sense, (16) contains the ultrametric condition for the partition elements.

Step 4. Selection of F (a covering of $B(x,r)$ by partition elements).

Fix $g \in L_\infty(E')$ and extend g to a function on X that is zero outside of E'. Assume that

$$||\mathbb{E}(|g| \, |\mathcal{F}_{(3k+i+1)m})||_{L_\infty(E')} = 1.$$

In particular, for all $F \in \mathcal{F}_{(3k+i+1)m}$ we have

$$\int_F |g| \, d\mu = \int_{F \cap E'} |g| \, d\mu \leq \mu(F \cap E') \leq \mu(F). \tag{19}$$

Let $r \in R_k^i$ and $x \in E'$. Define

$$F := \bigcup \{C \in \mathcal{P}_{(3k+i+1)m} \colon C \cap B(x,r) \neq \emptyset\} \in \mathcal{F}_{(3k+i+1)m}.$$

Recall that $\mathcal{P}_k$ is a partition of E'', $E'' = \{x \colon d(x, E') \leq D\}$, $R \subseteq (0,1] \subset (0, D]$, $D > 1$, and $r \leq 1$. So $B(x,r) \subseteq E''$, i.e. r is small enough such that $B(x,r)$ is covered by partition elements in the definition of F, so

$$F \supseteq B(x,r). \tag{20}$$

Recall, that we assumed $\operatorname{diam}(E'')=1$, and so by the definition of a partition tree, $\operatorname{diam}\mathcal{P}_k \le 2^{-k}$, so by the definition of F, and since $r\in R_k^i$ implies $r\ge 2^{-(3k+i)m}$,

$$F \subseteq B(x, r+\sup_{C\in\mathcal{P}_{(3k+i+1)m}} \operatorname{diam}(C)) \subseteq B(x, r+2^{-(3k+i+1)m}) \subseteq B(x,(1+2^{-m})r). \tag{21}$$

Recalling that $2^{-m}\le\beta = 1/(64n\log K) < 1/n$ and $x\in E'$,

$$\begin{aligned}\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|g| &\overset{(20)}{\le} \frac{1}{\mu(B(x,r))}\int_F |g| \overset{(19)}{\le} \frac{\mu(F)}{\mu(B(x,r))} \\ &\overset{(21)}{\le} \frac{\mu(B(x,(1+2^{-m})r))}{\mu(B(x,r))} \le \frac{\mu(B(x,(1+1/n)r))}{\mu(B(x,r))} \overset{\text{Def.1.7}}{\le} K.\end{aligned} \tag{22}$$

Step 5. Applying the generalized Doob inequality.

We use Theorem 2.2 for the increasing sequence of σ-algebras $\{\mathcal{F}_{(3k+i+1)m}\}_{k\ge 0}$ and the sublinear operators $\{\widetilde{M}_{R_k^i}\}_{k\ge0}$, with $A = v^{1/p}Q$ (via (15)), with $B=K$ (via (22)), with $X := E'$ and with the localization property verified in (16). (Note that $\widetilde{E}_k^i \subset E'$, so $\widetilde{M}_{R_k^i} g$ is supported in E'.) So, Theorem 2.2 and the definition of v give

$$\begin{aligned}\mu(\sup_{k\ge0}\widetilde{M}_{R_k^i} f > 1) &\le (2^p v Q^p + 2^p K^p)\int_X |f|^p\,d\mu \\ &\lesssim \left(2^p\left(1+\frac{\log\log K}{1+\log n}\right)Q^p + 2^pK^p\right)\int_X |f|^p\,d\mu.\end{aligned}$$

Finally, (14) and (11) complete our proof

$$[\mu(M_R f>1)]^{1/p} \lesssim \left[\left(1+\frac{\log\log K}{1+\log n}\right)^{1/p} Q + K\right] ||f||_{L_p(X)}.$$

□

5. Proof of Corollary 1.12

Proof. Fix $m\in\mathbb{Z}_{>0}$, $f\in L_p(X)$, $r,\lambda>0$. Then

$$\begin{aligned}\mu(M_{R\cap[r,nr]}f>\lambda) &= \mu(\max_{0\le j\le m-1} M_{R\cap[rn^{j/m},rn^{(j+1)/m}]}f>\lambda) \\ &\le \sum_{k=0}^{m-1}\mu(M_{R\cap[rn^{j/m},rn^{(j+1)/m}]}f>\lambda) \le m\cdot\max_{0\le j\le m-1}\mu(M_{R\cap[rn^{j/m},rn^{(j+1)/m}]}f>\lambda).\end{aligned}$$

So, using $p=1$ in Theorem 1.11, for every $m\in\mathbb{Z}_{>0}$ we have

$$||M_R||_{L_1(X)\to L_{1,\infty}(X)} \lesssim K + m\left(1+\frac{\log\log K}{1+\log n}\right)^{1/p}\cdot\sup_{r>0}\left|\left|M_{R\cap[r,n^{1/m}r]}\right|\right|_{L_p(X)\to L_{p,\infty}(X)}. \tag{23}$$

Let $m\ge 2n\log n$, so that $n^{1/m}\le 1+1/n$ (using $\log(n+1)-\log n = \int_n^{n+1}(1/y)dy$), and for all $r>0$,

$$M_{R\cap[r,n^{1/m}r]}f \le \frac{1}{\mu(B(x,r))}\int_{B(x,(1+1/n)r)}|f|\,d\mu \overset{\text{Def.1.7}}{\le} K\cdot A_{1+1/n}f, \tag{24}$$

where A_r is the averaging operator defined by

$$A_r f(x) := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| \, d\mu. \tag{25}$$

By Definition 1.8, $||A_{1+1/n} f||_1 \leq K \, ||f||_1$. Combining this with (23) and (24) gives

$$||M||_{1\to 1,\infty} \lesssim_K n \log n.$$

Now, let $m \asymp \log n$. Then $n^{1/m} \asymp 1$, so setting $R = 2^{\mathbb{Z}}$, we see that $R \cap [r, n^{1/m} r]$ contains a bounded number of points. Therefore, (23) shows that

$$||M_{2^{\mathbb{Z}}}||_{1\to 1,\infty} \lesssim_K \log n.$$

Similarly, by taking an even sparser sequence, one can get a smaller constant. □

References

[B] J. Bourgain, *On the L_p-bounds for maximal functions associated to convex bodies in $\mathbb{R}^n$*. Israel J. Math. 54 (1986), no. 3, 257265.

[M] D. Müller, *A geometric bound for maximal functions associated to convex bodies.* Pacific J. Math. 142 (1990), no. 2, 297312.

[NT] A. Naor, and T. Tao, Random martingales and localization of maximal inequalities. J. Funct. Anal. 259 (2010), no. 3, 731779.

[S1] E. M. Stein, Topics in harmonic analysis related to the Littlewood-Paley theory. Annals of Mathematics Studies, No. 63 Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo 1970.

[S2] E. M. Stein, Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970.

[S3] E. M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. With the assistance of Timothy S. Murphy. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.

[SS] E. M. Stein and J.-O. Strömberg, *Behavior of maximal functions in $\mathbb{R}^n$ for large n.* Ark. Mat. 21 (1983), no. 2, 259269.