STRONG CONTRACTION AND INFLUENCES IN TAIL SPACES

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Abstract. We study contraction under a Markov semi-group and influence bounds for functions all of whose low level Fourier coefficients vanish. This study is motivated by the explicit construction of 3-regular expander graphs of Mendel and Naor, though our results have no direct implication for the construction of expander graphs. In the positive direction we prove an L_p Poincaré inequality and moment decay estimates for mean 0 functions and for all $1 < p < \infty$, proving the degree one case of a conjecture of Mendel and Naor. In the negative direction, we answer negatively two questions of Hatami and Kalai concerning extensions of the Kahn-Kalai-Linial and Harper Theorems to tail spaces. For example, we construct a function $f\colon \{-1,1\}^n \to \{-1,1\}$ whose Fourier coefficients vanish up to level $c\log n$, with all influences bounded by $C\log n/n$ for some constants $0 < c, C < \infty$. That is, the Kahn-Kalai-Linial Theorem cannot be improved, even if we assume that the first $c\log n$ Fourier coefficients of the function vanish. This implies there is a phase transition in the largest guaranteed influence of functions $f\colon \{-1,1\}^n \to \{-1,1\}$, which occurs when the first $g(n)\log n$ Fourier coefficients vanish and $g(n)\to\infty$ as $n\to\infty$ or $g(n)$ is bounded as $n\to\infty$.

1. Introduction

Let Ω be a finite set, and let μ be a probability measure on Ω. Let $\mathbb{E}$ denote the expectation with respect to the measure μ. Let $f\colon \Omega\to\mathbb{R}$ and for any $1\leq p<\infty$, let $\|f\|_p := (\mathbb{E}\,|f|^p)^{1/p}$, with $\|f\|_\infty = \inf\{\lambda>0\colon \mu(f^{-1}((-\infty,-\lambda)\cup(\lambda,+\infty)))=0\}$.

Assumption 1. Let $(P_t)_{t\geq 0}$ be a symmetric Markov semigroup[1] on $L_2(\Omega,\mu)$, with generator $L = -\frac{d}{dt}P_t\big|_{t=0^+}$. Assume that L satisfies the L_2 Poincaré inequality with constant $C>0$:

$$\mathbb{E}f^2 - (\mathbb{E}f)^2 \leq C\cdot \mathbb{E}fLf, \quad \forall\, f\colon\Omega\to\mathbb{R}, \tag{1}$$

or equivalently[2], $\|P_tf\|_2 \leq e^{-t/C}\|f\|_2$, for all $t\geq 0$ and every $f\colon \Omega\to\mathbb{R}$ with $\mathbb{E}f=0$.

Remark 1.1. Note that the smaller C is, the better the estimates become. (C is the inverse of the spectral gap.)

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[1]That is, for all $t,s>0$, for all $f,g\colon\Omega\to\mathbb{R}$, for all $1\leq p\leq\infty$, we have: $P_t\colon L_2(\Omega,\mu)\to L_2(\Omega,\mu)$ is linear, $P_{t+s}=P_tP_s$, $\|P_tf\|_p\leq\|f\|_p$, $P_0=\mathrm{Id}$, $P_tf\geq 0$ if $f\geq 0$, $P_t1=1$, $\lim_{t\to 0^+}P_tf=f$ in $L_2(\Omega,\mu)$, and $\int(P_tf)gd\mu=\int f(P_tg)d\mu$.

[2]To see the equivalence, note that $(d/dt)\mathbb{E}(P_tf)^2 = 2\mathbb{E}P_tf(-L)P_tf$. So, assuming (1) and $\mathbb{E}f=0$, we have $(d/dt)[e^{2t/C}\mathbb{E}(P_tf)^2] = e^{2t/C}2[\mathbb{E}P_tfLP_tf + (1/C)\|P_tf\|_2]\leq 0$, so $\|P_tf\|_2\leq e^{t/C}$. Conversely, assuming $\|P_tf\|_2\leq e^{-t/C}\|f\|_2$ and $\mathbb{E}f=0$, we have $(d/dt)[e^{2t/C}\mathbb{E}(P_tf)^2]\leq 0$, so $\mathbb{E}P_tfLP_tf + (1/C)\|P_tf\|_2$, then letting $t\to 0^+$ gives (1).

Theorem 1.2 (**Main Result; Heat Smoothing** [HMO14])**.** *Suppose Assumption 1 holds. Then, for every $p \in (1, \infty)$ and every $f : \Omega \to \mathbb{R}$ with $\mathbb{E}f = 0$, for every $t > 0$,*

$$\|P_t f\|_p \leq \exp\left(-\frac{(2p-2)t}{(p^2-2p+2)C}\right) \cdot \|f\|_p.$$

Remark 1.3. For $\Omega = \mathbb{R}$, $p = 4$ and $d\mu = e^{-x^2/2}dx/\sqrt{2\pi}$, Theorem 1.2 was proven by P. Cattiaux, as noted in [MN14].

Remark 1.4. Theorem 1.2 solves the degree-one case of a more general conjecture of Mendel-Naor [MN14, Remark 5.5]; see Conjecture 1 below.

Theorem 1.2 is equivalent to the following statement, which can be seen by differentiating Theorem 1.2 in t:

Theorem 1.5 (**Poincaré Inequality** [HMO14])**.** *Suppose Assumption 1 holds. Then for every $p \in (1, \infty)$ and every $f\colon \Omega \to \mathbb{R}$ with $\mathbb{E}f = 0$,*

$$\mathbb{E}\,|f|^{p-1}\,\mathrm{sign}(f) Lf \geq \frac{2p-2}{(p^2-2p+2)C} \cdot \mathbb{E}\,|f|^p\,.$$

Remark 1.6. These results can be extended without difficulty to infinite spaces.

Example 1.7. Let n be a positive integer. Let μ denote uniform probability measure on $\{-1,1\}^n$. Any $f\colon \{-1,1\}^n \to \mathbb{R}$ can be written as $f = \sum_{S\subseteq\{1,\ldots,n\}} \widehat{f}(S)W_S$, where for all $x = (x_1, \ldots, x_n) \in \{-1,1\}^n$, $W_S(x) := \prod_{i\in S} x_i$ and $\widehat{f}(S) := 2^{-n}\sum_{x\in\{-1,1\}^n} f(x)W_S(x)$. For any $t \geq 0$, define $P_t f := \sum_{S\subseteq\{1,\ldots,n\}} e^{-t|S|}\widehat{f}(S)W_S$, and define $Lf := \sum_{S\subseteq\{1,\ldots,n\}} |S|\,\widehat{f}(S)W_S$. Then Assumption 1 holds with $C = 1$, since

$$\mathbb{E}f^2 - (\mathbb{E}f)^2 = \sum_{S\subseteq\{1,\ldots,n\}\colon\, |S|\geq 1} |\widehat{f}(S)|^2 \leq \sum_{S\subseteq\{1,\ldots,n\}\colon\, |S|\geq 1} |S|\,|\widehat{f}(S)|^2 = \mathbb{E}fLf.$$

2. Motivation and Background

Our general motivation for the investigation of Theorems 1.2 and 1.5 comes from understanding the smoothing properties of semigroups, and obtaining sharp quantitative bounds. Smoothing of semigroups is reflected e.g. in the **hypercontractive inequality** [Bon70, Nel73, Gro75]. Returning to Example 1.7, we have,

$$\|P_t f\|_q \leq \|f\|_p \quad \forall\, 1 \leq p \leq q,\, \forall\, t \geq \frac{1}{2}\log\left(\frac{q-1}{p-1}\right),\, \forall\, f\colon \{-1,1\}^n \to \mathbb{R}.$$

We now return to the case of general semigroups. It is natural to consider the relation of Theorem 1.2 to the hypercontractive inequality itself.

Question 2.1. Is it true that, if Assumption 1 is satisfied , then the hypercontractive inequality holds:

$$\|P_t f\|_{1+e^{\beta t}} \leq \|f\|_2 \quad \forall\, t > 0,$$

where $\beta > 0$ depends only on C?

Answer. No [DSC96, Lemma 4.2]. The largest possible constant β in the hypercontractive inequality for any finite k-regular, undirected[3] graph $G = (V, E)$ (where L is the generator of the simple random walk on G, i.e. L is the standard (un-normalized) Laplacian $Lf(x) = (1/2)\sum_{y:\,(x,y)\in E}(f(x) - f(y))$.) satisfies

$$\beta \leq 10k(\log k)\frac{(10 + \log\log|V|)}{2\log(|V|/4)}.$$

However, there exists a sequence of 3-regular graphs $(V_n, E_n)_{n=1}^{\infty}$ such that $|V_n| \to \infty$ as $n \to \infty$, while Assumption 1 holds with constant $C_n < C < \infty$ for all $n \geq 1$. That is, there exist sequences of 3-regular **expander graphs**. Note then that the hypercontractive constant satisfies $\beta_n \to 0$ as $n \to \infty$.

Remark 2.2. In summary: in general, one cannot deduce hypercontractivity from having an L_2 spectral gap. However, one can deduce an L_2 spectral gap from hypercontractivity [DSC96, Lemma 3.1].

In their study of a general notion of expander[4] (with respect to all uniformly convex spaces), Mendel and Naor made the following conjecture:

Conjecture 1 (**Heat Smoothing**). [MN14, Remark 5.5] *Let* $1 < p < \infty$. *Let* $f\colon \{-1,1\}^n \to \mathbb{R}$ *with* $\mathbb{E} f W_S = 0$ *for all* $S \subseteq \{1, \ldots, n\}$ *with* $|S| < k$. *Then*

$$\forall t > 0, \quad \|P_t f\|_p \leq e^{-tkc(p)}\|f\|_p. \tag{2}$$

Remark 2.3. If this conjecture were true (for functions valued in uniformly convex Banach spaces), then the expander graph construction from [MN14] would be greatly simplified.

By combining the hypercontractive inequality and Hölder's inequality, Mendel-Naor obtained the following weaker bound , which they attributed to P. A. Meyer [Mey84].

Theorem 2.4. [MN14, Lemma 5.4] *Let* $2 \leq p < \infty$. *Then there exists* $c(p) > 0$ *such that the following holds. Let* $f\colon \{-1,1\}^n \to \mathbb{R}$ *with* $\mathbb{E} f W_S = 0$ *for all* $|S| < k$. *Then*

$$\forall\, t > 0, \quad \|P_t f\|_p \leq e^{-k\min(t,t^2)c(p)}\|f\|_p.$$

$$\|Lf\|_p \geq c(p)\sqrt{k}\,\|f\|_p.$$

The second inequality can be considered a "higher-order" Poincaré inequality, and it follows from the first by writing $f = \int_0^\infty e^{-tL} Lf\,dt$ and then applying the $L_p(\{-1,1\}^n)$ triangle inequality.

Recall that we proved

[3] A graph $G = (V, E)$ is a finite set V of vertices and a multi-subset $E \subseteq V \times V$, so that some pair $(u, v) \in E$ is allowed to appear multiple times. We always impose the condition $(u, v) \in E \implies (v, u) \in E$. We denote $E(u, v) = E(v, u)$ as the number of times that (u, v) appears in E. The degree of $u \in V$ is $\sum_{v\in V} E(u, v)$. So, a self-loop contributes 1 to the degree of a vertex. Recall that an n-vertex graph $G = (V, E)$ is k-regular if every $u \in V$ has degree k.

[4] Recall that we know that constant degree expander graphs exist by the probabilistic method [HLW06]. Their explicit construction is more difficult. The first such construction goes back to [Mar73]. Since then, other constructions were found using algebraic methods. More recently, a purely combinatorial construction was found [RVW02]. This construction was one ingredient in [MN14].

Theorem 2.5 (**Poincaré Inequality** [HMO14])**.** *Under the above assumptions for every* $p \in (1,\infty)$ *and every* $f : \{-1,1\}^n \to \mathbb{R}$ *with* $\mathbb{E}f = 0$ *there is*

$$\mathbb{E}\,|f|^{p-1}\,\mathrm{sign}(f)Lf \geq \frac{2p-2}{(p^2-2p+2)} \cdot \mathbb{E}\,|f|^p\,.$$

The usual Poincaré inequality corresponds to the case $p = 2$ of Theorem 2.5. Theorem 2.5 should be contrasted with Beckner's Poincaré inequality.

Theorem 2.6. [Bec89] *Let* $f\colon \{-1,1\}^n \to \mathbb{R}$*. For all* $1 \leq p \leq 2$*,*

$$(2-p)\mathbb{E}fLf \geq \mathbb{E}\,|f|^2 - (\mathbb{E}\,|f|^p)^{2/p}.$$

Specifically, Beckner notes that, for $t > 0$ with $e^{-2t} = p-1$, $(2-p)\mathbb{E}fLf \geq \mathbb{E}\,|f|^2 - \mathbb{E}\,|P_t f|^2$ by Fourier analysis. He then adds the hypercontractive inequality [Bon70, Nel73, Gro75] to this inequality to prove Theorem 2.6. However, Theorem 1.5 does not seem to follow from hypercontractivity so we need to apply different methods.

3. Hypercontractivity, Talagrand's Inequality, and Kahn-Kalai-Linial

We recall some standard definitions.

Definition 3.1 (**Influences**)**.** Let $f\colon \{-1,1\}^n \to \{-1,1\}$ and let $i \in \{1,\ldots,n\}$. Define the i'th *influence* $I_i(f) \in \mathbb{R}$ of a function $f : \{-1,1\}^n \to \{-1,1\}$ by

$$I_i(f) := P[f(x_1,\ldots,x_n) \neq f(x_1,\ldots,x_{i-1},-x_i,x_{i+1},\ldots,x_n)],$$

where x_i are i.i.d. uniform random variables on $\{-1,1\}$ for all $i = 1,\ldots,n$. Equivalently,

$$I_i f = \sum_{S\subseteq\{1,\ldots,n\}\colon\, i\in S} (\widehat{f}(S))^2 = \|\partial_i f\|_2^2\,,$$

where $\partial_i f = [f(x) - f(x_1,\ldots,x_{i-1},-x_i,x_{i+1},\ldots,x_n)]/2$.

Remark 3.2 (Voting Interpretation)**.** Let $f\colon \{-1,1\}^n \to \{-1,1\}$. We think of $x = (x_1,\ldots,x_n) \in \{-1,1\}^n$, where x_i is the vote of person $i \in \{1,\ldots,n\}$ for the candidate $x_i \in \{-1,1\}$. Then $f(x)$ is the winner of the election, given the votes x. And $I_i f$ is the probability that person $i \in \{1,\ldots,n\}$ can change the outcome of the election.

Example 3.3. The majority function is defined for $f\colon \{-1,1\}^n \to \{-1,1\}$ (with n odd) by the formula

$$f(x_1,\ldots,x_n) = \mathrm{sign}(x_1 + \cdots + x_n).$$

This function satisfies[5] $I_i f \sim \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{n-1}}$ as $n \to \infty$.

So your influence in an election is much larger than you think!

Question 3.4. What is the smallest possible value of $\max_{i=1,\ldots,n} I_i f$?

Note that, it is easy to have $I_1 f = 1$. Consider $f(x_1,\ldots,x_n) = x_1$. So, it is easy to construct a function with one large influence.

Proposition 3.5. [BOL89, Theorem 3] There exists a universal constant $c' > 0$ and there exists a Boolean function $f\colon \{-1,1\}^n \to \{-1,1\}$ with $\mathbb{E}f = 0$ such $\max_{i=1,\ldots,n} I_i(f) \leq c'(\log n)/n$.

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[5] $I_i f = \binom{n-1}{(n-1)/2}2^{-n} = \frac{(n-1)!}{([(n-1)/2]!)^2}2^{-n} \sim \frac{\sqrt{2\pi(n-1)}((n-1)/e)^{n-1}}{2\pi((n-1)/2)((n-1)/(2e))^{n-1}}2^{-n} \sim \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{n-1}}$.

Remark 3.6. The function constructed here is known as the tribes function. Suppose $n = br$, $b, r \in \mathbb{N}$. We write $\{-1,1\}^n = (\{-1,1\}^r)^b$. Let $g_1: \{-1,1\}^r \to \{-1,1\}$ denote a unanimous vote for candidate 1. That is, $g_1(1,\ldots,1) = 1$, and $g_1(x) = -1$ otherwise. Let $g_2: \{-1,1\}^b \to \{-1,1\}$ denote a unanimous vote for candidate -1. That is, $g_{-1}(-1,\ldots,-1) = -1$ and $g_{-1}(x) = 1$ otherwise. The tribes function is given by

$$f(x) = g_{-1}(g_1(x_1,\ldots,x_r), g_1(x_{r+1},\ldots,x_{2r}),\ldots,g_1(x_{(b-1)r+1},\ldots,x_{br})).$$

That is, we have b tribes of size r each, and each tribe conducts a unanimous vote for candidate 1, and the result of these tribal elections is input to a unanimous vote for candidate -1. By choosing the constants b, r correctly ($r \approx \log n - \log\log n$, $b \approx n/\log n$), f will have mean zero. Also, $I_i f \approx (\log n)/n$ for all $i \in \{1,\ldots,n\}$.[6]

Kahn, Kalai and Linial then showed that the influence bound in Proposition 3.5 is in fact the best possible, thereby proving the conjecture of Ben-Or and Linial.

Theorem 3.7 (Kahn-Kalai-Linial). [KKL88, Theorem 3.1] *There exists a universal constant $c > 0$ such that, for any $f: \{-1,1\}^n \to \{-1,1\}$, $\max_{i=1,\ldots,n} I_i(f) \geq c(\mathbb{E}(f-\mathbb{E}f)^2)(\log n)/n$.*

Remark 3.8. This Theorem is proven using the hypercontractive inequality. The Kahn-Kalai-Linial Theorem is slightly generalized to Talagrand's inequality (which is also proven using hypercontractivity).

If a function $f: \{-1,1\}^n \to \{-1,1\}$ not only has mean zero, but it also has many Fourier coefficients which are zero, it similarly seems that even more special structure should exist within the Fourier coefficients of f. That is, perhaps this function should have a larger influence than a mean zero function. Hatami and Kalai therefore asked the following question, which would improve upon Theorem 3.7.

Question 3.9. Suppose $k = k(n) \to \infty$ as $n \to \infty$. Does there exist $\omega(k) > 0$ such that $\omega(k) \to \infty$ as $k \to \infty$, such that the following statement holds? Let $f: \{-1,1\}^n \to \{-1,1\}$ with $\mathbb{E} f W_S = 0$ for all $S \subseteq \{1,\ldots,n\}$ with $|S| \leq k$. Then $\max_{i=1,\ldots,n} I_i f \geq ((\log n)/n)\cdot\omega(k)$.

Hatami speculated that a positive answer to the question above may help in proving the Entropy Influence Conjecture. Here we prove that the answer to the question is negative by showing that

Theorem 3.10 (Question 3.9 for $k = \log n$)**.** *There exists $0 < C, c < \infty$ such that, for infinitely many $n \in \mathbb{N}$, there exists $f: \{-1,1\}^n \to \{-1,1\}$ with $\mathbb{E} f W_S = 0$ for all $S \subseteq \{1,\ldots,n\}$ with $|S| \leq c\log n$ such that $\max_{i=1,\ldots,n} I_i f \leq C(\log n)/n$.*

In other words, there is a phase transition for the maximum influence of Boolean functions with vanishing Fourier coefficients. This phase transition occurs when we require the first $k(n)$ Fourier coefficients to vanish where $k(n)/\log n$ is either bounded or unbounded, as $n \to \infty$. We note that the functions constructed in Theorem 3.10 do not provide a counter example to the Entropy Influence conjecture as their entropy is of the same order as for the standard Tribes function.

[6] Voter $i \in \{1,\ldots,n\}$ changes the outcome of the election if and only if: all other tribes satisfy $g_1(x_\ell,\ldots) = -1$, and all other $r-1$ people in the tribe of i have vote $x_j = 1$. Assuming a uniform distribution of votes, this scenario has probability $(1-2^{-r})^{b-1}2^{-(r-1)} = \frac{\log n}{n}(1+1-(1-2^{-r})^{b-1}) \approx \frac{\log n}{n}$.

Remark 3.11. To construct our function f, we modify the tribes function

$$g_{-1}(g_1(x_1,\ldots,x_r),g_1(x_{r+1},\ldots,x_{2r}),\ldots,g_1(x_{(b-1)r+1},\ldots,x_{br})).$$

We now replace g_1 with G, where $G(x_1,\ldots,x_r)=1$ if and only if $((1-x_1)/2,\ldots,(1-x_r)/2)\in C$ where C is a good linear code [Mac63]. G is chosen since it takes values in $\{-1,1\}$, but it has vanishing low level Fourier coefficients. Some extra adjustments are needed to ensure that $\mathbb{E}f=0$.

We also note that if $k=g(n)\log n$, where $g(n)\to\infty$ then it is trivial to improve the KKL estimate since, if $f\colon\{-1,1\}^n$, then $1=\|f\|_2^2=\sum_{S\subseteq\{1,\ldots,n\}}|\widehat{f}(S)|^2$, and

$$\sum_{i=1}^n I_i f=\sum_{S\subseteq\{1,\ldots,n\}}|S|\,|\widehat{f}(S)|^2\geq\sum_{S\subseteq\{1,\ldots,n\}}k|\widehat{f}(S)|^2=k$$

which implies $\max_{i=1,\ldots,n}I_i f\geq g(n)(\log n)/n$. When $g(n)\to\infty$, one can also improve Talagrand's inequality by essentially repeating its proof.

4. Appendix: Some Side Remarks

Remark 4.1. A sufficient condition for satisfying a particular logarithmic-Sobolev inequality, which (visually at least) resembles Theorem 1.2 is the following [Oll09, Theorem 45]:

$$\|P_t f\|_{Lip}\leq e^{-tc}\|f\|_{Lip}\,,\qquad\forall\,t>0,\,\forall f\colon\Omega\to\mathbb{R}.$$

Here d is a metric on Ω, and $\|f\|_{Lip}:=\sup_{x,y\in\Omega\colon x\neq y}\frac{|f(x)-f(y)|}{d(x,y)}$. (See also [Jou09].)

Remark 4.2. Diaconis and Saloff-Coste [DSC96] use a slightly different normalization than us. They define the spectral gap as the largest constant $\lambda>0$ such that

$$\frac{1}{\lambda}\sum_{x,y}|f(x)-f(y)|^2K(x,y)\pi(x)\geq\sum_{x,y}|f(x)-f(y)|^2\pi(y)\pi(x).$$

Where $K(x,y)$ is the probability transition kernel, $K(x,y)\geq 0$ and $\sum_y K(x,y)=1$, and where π is the invariant probability measure with respect to K. They then define $H_t=\exp(-t(I-K))$ to be the corresponding semigroup. They also define the Log-Sobolev constant as the largest constant $\alpha>0$ such that

$$\frac{1}{\alpha}\sum_{x,y}|f(x)-f(y)|^2K(x,y)\pi(x)\geq\sum_x|f(x)|^2\log(|f(x)|^2/\|f\|_2^2)\pi(x).$$

They show that α is equivalently defined as the largest constant β such that

$$\|H_t f\|_{1+e^{4\beta t}}\leq\|f\|_2\,,\quad\forall t>0.$$

Also, for example, with these definitions, the spectral gap of the simple random walk on $\{-1,1\}^n$ is $\lambda=2/n$, and the log-Sobolev constant is $\alpha=1/n$. To reconcile the difference of our constants, note that K is off by a factor of n. Specifically, if f is a mean zero function, then we have

$$\mathbb{E}_{x,y}(f(x)-f(y))^2=\mathbb{E}_{x,y}(f(x)^2-2f(x)f(y)+f(y)^2)=2\mathbb{E}f^2.$$

Define $h_i(x) = 1$ when $x = (1, \ldots, 1)$, $h_i(x) = -1$ when $x = (1, \ldots, 1, -1, 1, \ldots, 1)$, where the -1 occurs in the i^{th} entry, and $h_i(x) = 0$ for all other $x \in \{-1,1\}^n$. Define $f * g(x) = \mathbb{E}_y f(y)g(x-y)$. Note that $\widehat{f * g}(S) = \widehat{f}(S)\widehat{g}(S)$. Also, $f * h_i(x) = (f(x) - f(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n))/2$, and by inspection, $h_i = \sum_{S \subseteq \{1,\ldots,n\}\colon i \in S} W_S$. So, we have

$$\frac{1}{4} \sum_{(x,y) \in E} (f(x) - f(y))(g(x) - g(y)) = \frac{1}{4} 2^{-n} \sum_{x \in V} \frac{1}{n} \sum_{y \sim x} (f(x) - f(y))(g(x) - g(y))$$

$$= \frac{1}{n} \sum_{i=1}^{n} \langle f * h_i, g * h_i \rangle = \frac{1}{n} \sum_{i=1}^{n} \sum_{S \subseteq \{1,\ldots,n\}\colon i \in S} \widehat{f}(S)\widehat{g}(S) = \frac{1}{n} \sum_{S \subseteq \{1,\ldots,n\}} |S| \, \widehat{f}(S)\widehat{g}(S) = \frac{1}{n} \mathbb{E} f L g.$$

So, for k-regular graphs, Diaconis and Saloff-Coste divide their Laplacian by the degree, while we do not.

The original motivation for Theorem 1.2 was a simplified construction of expander graphs which have a spectral gap with respect to any uniformly convex Banach space (or more generally for any K-convex space) [MN14]. A Banach space X is said to be K-convex if there exists $\varepsilon_0 > 0$ and $n_0 \in \mathbb{N}$ such that any embedding of $\ell_1^{n_0}$ into X incurs bi-Lipschitz distortion at least $1 + \varepsilon_0$. The following theorem is the original motivation for Theorem 1.2 and the main result of [MN14], which strengthens and clarifies previous constructions of expander graphs.

Theorem 4.3. [MN14, Theorem 1.1] *There exists a sequence of* 3*-regular graphs* $\{G_n\}_{n \in \mathbb{N}} = \{(V_n, E_n)\}_{n \in \mathbb{N}}$ *with* $\lim_{n \to \infty} |V_n| = \infty$ *such that, for any K-convex Banach space X, there exists* $\gamma(X, \|\cdot\|_X^2) > 0$ *such that, for any* $F\colon V_n \to X$,

$$\frac{1}{|V_n|^2} \sum_{(u,v) \in V_n \times V_n} \|F(u) - F(v)\|_X^2 \leq \frac{\gamma(X, \|\cdot\|_X^2)}{3\,|V_n|} \sum_{(x,y) \in E_n} \|F(x) - F(y)\|_X^2 .$$

Remark 4.4. The construction of these graphs appears to be explicit, insofar as the construction of asymptotically good linear codes is explicit.

To prove Theorem 4.3, the authors proved an infinite family of estimates for functions $f\colon \{-1,1\}^n \to X$ where X is a K-convex Banach space. We now describe this family of inequalities, which will lead us to Theorem 1.2. Any $f\colon \{-1,1\}^n \to X$ can be written as $f = \sum_{S \subseteq \{1,\ldots,n\}} \widehat{f}(S) W_S$, where for all $x = (x_1, \ldots, x_n) \in \{-1,1\}^n$, $W_S(x) := \prod_{i \in S} x_i$ and $\widehat{f}(S) := 2^{-n} \sum_{x \in \{-1,1\}^n} f(x) W_S(x)$. For any $t \geq 0$, $p > 1$, define $P_t f := \sum_{S \subseteq \{1,\ldots,n\}} e^{-t|S|} \widehat{f}(S) W_S$, $\|f\|_p := (2^{-n} \sum_{x \in \{-1,1\}^n} \|f(x)\|_X^p)^{1/p}$. Define

$$\mathrm{Rad}(f) := \sum_{j=1}^{n} \widehat{f}(\{j\}) W_{\{j\}}, \qquad K(X) := \sup_{n \in \mathbb{N}} \|\mathrm{Rad}\|_{L_2(\{-1,1\}^n, X) \to L_2(\{-1,1\}^n, X)} .$$

The operator $\mathrm{Rad}(f)$ is known as the Rademacher projection, and $K(X)$ is referred to as the K-convexity constant of X. In particular, X is K-convex if and only if $K(X) < \infty$, as proven by Pisier [Pis82, Theorem 2.1].

By creating a quantitative proof of a deep theorem of Pisier's [Pis82, Theorem 1.2] concerning the holomorphic extension of the semigroup P_t, Mendel and Naor proved the following decay estimate for the semigroup P_t.

Theorem 4.5. [MN14, Theorem 5.1] *For every $K, p > 1$, there exist $A(K,p) \in (0,1)$ and $B(K,p), C(K,p) > 2$ such that, for every K-convex Banach space $(X, \|\cdot\|_X)$ with $K(X) \leq K$, for every $k, n \in \mathbb{N}$, for every $t > 0$ and for every $f\colon \{-1,1\}^n \to X$ with $\widehat{f}(S) = 0$ for all $S \subseteq \{1, \ldots, n\}$ with $|S| < k$,*

$$\|P_t f\|_p \leq C(K,p) \cdot \exp\left(-k \cdot A(K,p) \cdot \min(t, t^{B(K,p)})\right). \tag{3}$$

Theorem 4.5 is the crucial ingredient in a construction of a single sequence of 3-regular expander graphs, which simultaneously have a spectral gap with respect to all K-convex Banach spaces X [MN14, Theorem 1.1]. However, it is conjectured that the term $\min(t, t^{B(K,p)})$ in (3) should be able to be improved to t, since this is possible when $p = 2$ and $X = \ell_2$ [MN14, Remark 5.12]. If this conjecture were true, then the construction of the expander graphs of [MN14, Theorem 1.1] would be simplified and improved. However, even in the case $X = \mathbb{R}$, we do not know how to solve this conjecture. So, in this work, we focus on $X = \mathbb{R}$ and $k = 1$ in trying to improve Theorem 4.5.

Remark 4.6. Note that, even in the case $\Omega = \{-1,1\}^n$ with the metric induced from the ℓ_1 norm, the doubling constant of Ω is unbounded as $n \to \infty$. So, Theorem 1.2 does not seem to be provable by "transplantation" techniques, as in [TDOS02]. Indeed, the use in the proof of Theorem 4.5 of holomorphic extension of the semigroup P_t avoids the difficulties inherent in "transplantation" techniques.

Remark 4.7. One may believe that Theorem 1.2 could be proven by Littlewood-Paley Theory and/or multiplier theorems. However, such tools do not obtain the sharp constants of Theorem 1.2. Indeed, Theorem 4.5 could be equivalently described as an attempt to prove a general Littlewood-Paley inequality. (Recall that the crucial point in proving the Littlewood-Paley inequality is inverting the Laplacian on L_p spaces, and this follows exactly from Theorem 4.5 or Theorem 1.2)

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