LOW CORRELATION NOISE STABILITY OF EUCLIDEAN SETS

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Abstract. The noise stability of a Euclidean set is a well-studied quantity. This quantity uses the Ornstein-Uhlenbeck semigroup to generalize the Gaussian perimeter of a set. The noise stability of a set is large if two correlated Gaussian random vectors have a large probability of both being in the set. We will first survey old and new results for maximizing the noise stability of a set of fixed Gaussian measure. We will then discuss some recent results for maximizing the low-correlation noise stability of three sets of fixed Gaussian measures which partition Euclidean space. Finally, we discuss more recent results for maximizing the low-correlation noise stability of symmetric subsets of Euclidean space of fixed Gaussian measure. All of these problems are motivated by applications to theoretical computer science.

1. Gaussian Isoperimetry and Noise Stability

Definition 1.1. We call $H \subseteq \mathbb{R}^n$ a **half space** if H is a set of points lying on one side of a hyperplane.

Definition 1.2 (**Gaussian Measure**)**.** Let n be a positive integer. Let $A \subseteq \mathbb{R}^n$ be a measurable set. Define the *Gaussian measure of* A to be

$$\gamma_n(A) := \int_A e^{-(x_1^2+\cdots+x_n^2)/2} \frac{dx}{(2\pi)^{n/2}}.$$

Theorem 1.3 (**Gaussian Isoperimetric Inequality**, [SC74, Bor75])**.** *The half space has the smallest Gaussian surface area among all sets of fixed Gaussian measure. That is, let* $H \subseteq \mathbb{R}^n$ *be a half space such that* $\gamma_n(A) = \gamma_n(H)$*. Then*

$$\gamma_{n-1}(\partial A) \geq \gamma_{n-1}(\partial H),$$

where $\gamma_{n-1}(\partial A) := \liminf_{\varepsilon \to 0} \frac{1}{2\varepsilon} \gamma_n\{x \in \mathbb{R}^n : \exists\, a \in \partial A \text{ such that } \|x - a\|_2 < \varepsilon\}$.

Theorem 1.4 ([Bor85, Led94, MN15, Eld15])**.** *Among all subsets of Euclidean space* $\mathbb{R}^n$ *of fixed Gaussian measure, a half space maximizes noise stability (for positive correlation).*

In order to be more formal, we now define noise stability. Let $f : \mathbb{R}^n \to [-1, 1]$. We define T_ρ, the **Ornstein-Uhlebeck operator** with correlation $\rho \in (-1, 1)$, by

$$T_\rho f(x) := \int_{\mathbb{R}^n} f(x\rho + y\sqrt{1-\rho^2}) d\gamma_n(y), \qquad \forall\, x \in \mathbb{R}^n. \tag{1}$$

Definition 1.5 (**Noise Stability**)**.** Let n be a positive integer. Let $\rho \in (-1, 1)$. Let $A \subseteq \mathbb{R}^n$ be a measurable set. Define the **Noise Stability of** A **with correlation** ρ to be

$$\int_{\mathbb{R}^n} 1_A(x) T_\rho 1_A(x) d\gamma_n(x).$$

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Equivalently, let $X = (X_1, \ldots, X_n), Y = (Y_1, \ldots, Y_n)$ be jointly normal standard n-dimensional Gaussian random vectors such that the covariances satisfy $\mathbb{E}(X_i Y_j) = \rho \cdot 1_{\{i=j\}}$. Then the noise stability of A is

$$\mathbb{P}((X, Y) \in A \times A).$$

To see that both definitions are the same, note that

$$\int_{\mathbb{R}^n} 1_A(x) T_\rho(x) 1_A d\gamma_n(x) = \int_{\mathbb{R}^n} 1_A(x) \int_{\mathbb{R}^n} 1_A(x\rho + y\sqrt{1-\rho^2}) d\gamma_n(y) d\gamma_n(x)$$

$$=_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_A(x) 1_A(x\rho + y\sqrt{1-\rho^2}) d\gamma_n(y) d\gamma_n(x) = \mathbb{P}((X, Y) \in A \times A).$$

Remark 1.6. Theorem 1.4 can be proven by symmetrization [Bor85], by heat flow methods [Led94], [MN15], and by stochastic calculus methods [Eld15]. The latter two papers are prove stability estimates for Theorem 1.4. That is, [MN15] and [Eld15] show that A is close to a half space if and only if the noise stability of A is close to the maximum possible.

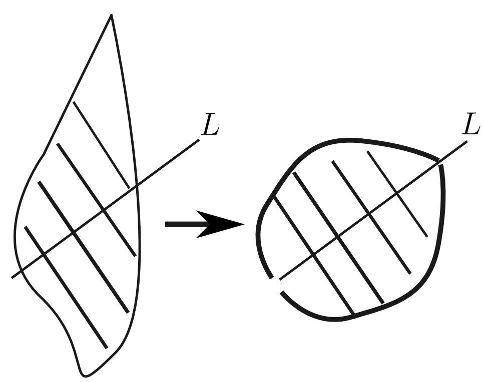

FIGURE 1. Classical Symmetrization

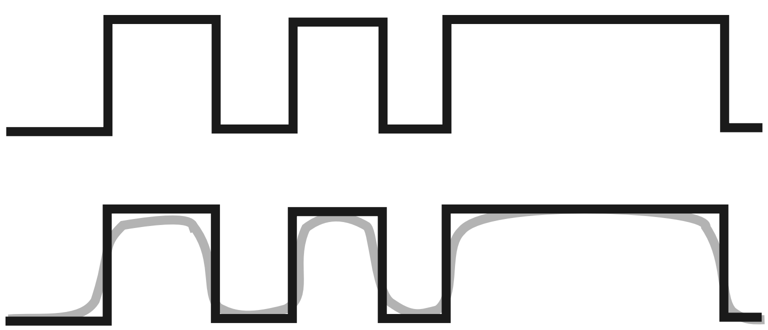

FIGURE 2. Depiction of heat evolution of the indicator function of a set

Remark 1.7. As $\rho \to 1$, noise stability (appropriately normalized) converges to Gaussian surface area [Led96, Proposition 8.5].

1.1. **Applications of Theorem 1.4.** Theorem 1.4 has a few important consequences. For example, it can be used to prove the following result from social choice theory

Theorem 1.8 (**Majority is Stablest Theorem, Informal, Probabilistic**, [MOO10])**.** *Among all voting methods for two candidates where each candidate has an equal chance of winning, and every person has a small influence over the outcome of the election, the majority function is the most noise stable.*

Here a set of votes between two candidates is a vector $x = (x_1, \ldots, x_n) \in \{-1, 1\}^n$, so that x_i is the vote of person i for candidate $x_i \in \{-1, 1\}$, for every $i \in \{1, \ldots, n\}$. And a voting method is a function $f\colon \{-1, 1\}^n \to \{-1, 1\}$, whose input is the set of votes x. And the output $f(x)$ is the winner of the election.

One of the two main ingredients in Theorem 1.8 is Theorem 1.4.

Another consequence of Theorem 1.4 is sharp computational hardness for the MAX-CUT problem, assuming the Unique Games Conjecture [KKMO07]. The MAX-CUT problem asks for the partition of the vertices of an undirected graph into two sets S and S^c that maximizes the number of edges that go from S to S^c. This problem is NP-hard, but we know how to find a cut of a graph which cuts about .878567 times the maximum possible number of cut edges in polynomial time. This number .878567 is the best possible approximation we can get in polynomial time, assuming the Unique Games Conjecture. The proof of this result uses Theorem 1.4.

2. Noise Stability for Multiple Sets

Theorem 1.4 can be restated in the following way.

Theorem 2.1 ([Bor85, Led94, MN15, Eld15])**.** *Among all partitions A_1, A_2 of Euclidean space $\mathbb{R}^n$ of fixed Gaussian measure, two opposing half spaces maximize the sum of the noise stabilities of A_1 and A_2.*

To see that Theorem 2.1 is equivalent to Theorem 1.4, note that if $A_1 \cup A_2 = \mathbb{R}^n$ and $A_1 \cap A_2 = \emptyset$, then $A_1 = A_2^c$, so $1_{A_2} = 1 - 1_{A_1}$, so

$$\sum_{i=1}^{2} \int_{\mathbb{R}^n} 1_{A_i}(x) T_\rho 1_{A_i}(x) d\gamma_n(x) = \int_{\mathbb{R}^n} 1_{A_1}(x) T_\rho 1_{A_1}(x) d\gamma_n(x) + \int_{\mathbb{R}^n} (1 - 1_{A_1}(x)) T_\rho (1 - 1_{A_1}(x)) d\gamma_n(x)$$
$$= 2 \int_{\mathbb{R}^n} 1_{A_1}(x) T_\rho 1_{A_1}(x) d\gamma_n(x) - 2\gamma_n(A) + 1.$$

Theorem 2.1 then has a natural generalization to more than two sets. For simplicity, we only mention the case of three sets of equal measure

Conjecture 1 (**Standard Simplex Conjecture for Three Sets**, [IM12])**.** *Let $n \geq 2$. Let (A_1, A_2, A_3) be a partition of $\mathbb{R}^n$. Suppose $\gamma_n(A_1) = \gamma_n(A_2) = \gamma_n(A_3) = 1/3$. Let B_1, B_2, B_3 be a partition of $\mathbb{R}^n$ into three 120 degree sectors. Then for all $\rho \in (0, 1)$,*

$$\sum_{i=1}^{3} \int_{\mathbb{R}^n} 1_{A_i}(x) T_\rho 1_{A_i}(x) d\gamma_n(x) \leq \sum_{i=1}^{3} \int_{\mathbb{R}^n} 1_{B_i}(x) T_\rho 1_{B_i}(x) d\gamma_n(x).$$

Remark 2.2. For applications, the most interesting case occurs when $\rho < 0$.

Theorem 2.3. [Hei14] *Conjecture 1 holds for $0 < \rho < \rho(n)$.*

Remark 2.4. The proof proceeds roughly as follows. The case $\rho = 0$ is relatively easy to understand. We can relate noise stability for ρ near zero to $\rho = 0$ using geometry, Hermite-Fourier, and fairly precise estimates for the Gaussian (Mehler) heat kernel. In particular, analytic methods are emphasized.

2.1. **Applications.** One consequence of Conjecture 1 is the following statement from social choice theorem.

Conjecture 2 (**Plurality is Stablest Conjecture, 3 Candidates, Informal**, [IM12])**.** *Among all voting methods where each of three candidates has an equal chance of winning, and every person has a small influence over the outcome of the election, the plurality function is the most noise stable voting method.*

Another consequence of Conjecture 1 is sharp computational hardness for the MAX-3-CUT problem, assuming the Unique Games Conjecture [KKMO07]. The MAX-3-CUT problem asks for the partition of the vertices of an undirected graph into three sets that maximizes the number of edges that go between the sets. This problem is NP-hard, but we know how to find a cut of a graph which cuts a constant fraction times the maximum possible number of cut edges in polynomial time. This fraction is the best possible approximation we can get for MAX-3-CUT in polynomial time, assuming the Unique Games Conjecture. The proof of this result uses Conjecture 1.

3. Noise Stability of Symmetric Sets

Another way to modify Theorem 1.4 is to add the restriction that the Euclean set A in question is symmetric. We say that a set $A \subseteq \mathbb{R}^n$ is symmetric if $A = -A$.

Conjecture 3 (Symmetric Gaussian Problem, Informal, [Bar01, CR11, O'D12])**.** *Among all symmetric subsets of $\mathbb{R}^n$ of fixed Gaussian measure, the ball centered at the origin or its complement maximizes noise stability.*

Theorem 3.1. [Hei15] $\exists\ \rho_0 > 0$ *such that,* $\forall\ \rho \in (-\rho_0, \rho_0)$ *and* $n = 1$, *Conjecture 3 holds.*

Theorem 3.2. [Hei15] *For any $n \geq 2$, there exists a measure restriction $0 < a < 1$ such that the ball centered at the origin and the complement of a ball centered at the origin, both of Gaussian measure a, do not maximize noise stability. That is, Conjecture 3 is false when $n \geq 2$.*

Remark 3.3. The proof of the first result proceeds along the same lines as Theorem 2.3, though the present result ends up being much simpler. The second result uses a second variation formula from [CS07]. In particular, analytic and variational methods are emphasized.

Remark 3.4. In Theorem 1.4, note that when a half space is translated, then the translated set still maximizes noise stability (for a different measure constraint). This property is crucial for the proofs of Theorem 1.4. On the other hand, Conjecture 3 has no such translation invariance property. And it turns out that Conjecture 1 also does not have such a translation invariance property [HMN15]. So, proofs of Conjecture 1 and Conjecture 3 must proceed in a different way than proofs of Theorem 1.4.

3.1. **Applications.** Conjecture 3, and weaker variants of it, are used in studying the communication complexity of the Gap-Hamming-Distance problem [CR11]. This analysis then leads to memory lower bounds for various algorithms.

References

[Bar01] Franck Barthe, *An isoperimetric result for the gaussian measure and unconditional sets*, Bulletin of the London Mathematical Society **33** (2001), 408–416.

[Bor75] Christer Borell, *The Brunn-Minkowski inequality in Gauss space*, Invent. Math. **30** (1975), no. 2, 207–216. MR 0399402 (53 #3246)

[Bor85] ———, *Geometric bounds on the Ornstein-Uhlenbeck velocity process*, Z. Wahrsch. Verw. Gebiete **70** (1985), no. 1, 1–13. MR 795785 (87k:60103)

[CR11] Amit Chakrabarti and Oded Regev, *An optimal lower bound on the communication complexity of gap Hamming distance*, Proc. 43rd Annual ACM Symposium on the Theory of Computing, 2011, pp. 51–60.

[CS07] Rustum Choksi and Peter Sternberg, *On the first and second variations of a nonlocal isoperimetric problem*, J. Reine Angew. Math. **611** (2007), 75–108. MR 2360604 (2008j:49062)

[Eld15] Ronen Eldan, *A two-sided estimate for the gaussian noise stability deficit*, Inventiones mathematicae **201** (2015), no. 2, 561–624 (English).

[Hei14] Steven Heilman, *Euclidean partitions optimizing noise stability*, Electron. J. Probab. **19** (2014), no. 71, 37. MR 3256871

[Hei15] ———, *Low correlation noise stability of symmetric sets*, preprint, arXiv:1511.00382, 2015.

[HMN15] Steven Heilman, Elchanan Mossel, and Joe Neeman, *Standard simplices and pluralities are not the most noise stable*, to appear, Israel Journal of Math, arXiv:1403.0885, 2015.

[IM12] Marcus Isaksson and Elchanan Mossel, *Maximally stable Gaussian partitions with discrete applications*, Israel J. Math. **189** (2012), 347–396. MR 2931402

[KKMO07] Subhash Khot, Guy Kindler, Elchanan Mossel, and Ryan O'Donnell, *Optimal inapproximability results for MAX-CUT and other 2-variable CSPs?*, SIAM J. Comput. **37** (2007), no. 1, 319–357. MR 2306295 (2008d:68035)

[Led94] Michel Ledoux, *Semigroup proofs of the isoperimetric inequality in Euclidean and Gauss space*, Bull. Sci. Math. **118** (1994), no. 6, 485–510. MR 1309086 (96c:49061)

[Led96] ———, *Isoperimetry and Gaussian analysis*, Lectures on probability theory and statistics (Saint-Flour, 1994), Lecture Notes in Math., vol. 1648, Springer, Berlin, 1996, pp. 165–294. MR 1600888 (99h:60002)

[MN15] Elchanan Mossel and Joe Neeman, *Robust optimality of Gaussian noise stability*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 2, 433–482. MR 3317748

[MOO10] Elchanan Mossel, Ryan O'Donnell, and Krzysztof Oleszkiewicz, *Noise stability of functions with low influences: invariance and optimality*, Ann. of Math. (2) **171** (2010), no. 1, 295–341. MR 2630040 (2012a:60091)

[O'D12] Ryan O'Donnell, *Open problems in analysis of boolean functions*, Preprint, arXiv:1204.6447v1, 2012.

[SC74] V. N. Sudakov and B. S. Cirel′son, *Extremal properties of half-spaces for spherically invariant measures*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **41** (1974), 14–24, 165, Problems in the theory of probability distributions, II. MR 0365680 (51 #1932)

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