The Propeller Conjecture in $\mathbb{R}^3$

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Let A_1, A_2, A_3 partition the plane.

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The direction of $z_i \in \mathbb{R}^2$ points to "where the mass is."

The length of z_i says "how much mass is there" and "how far the mass is from the origin."

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Among all partitions, **maximize**

$$||z_1||_2^2 + ||z_2||_2^2 + ||z_3||_2^2$$

Which partition "**pushes away**" the moment vectors from the origin the most?

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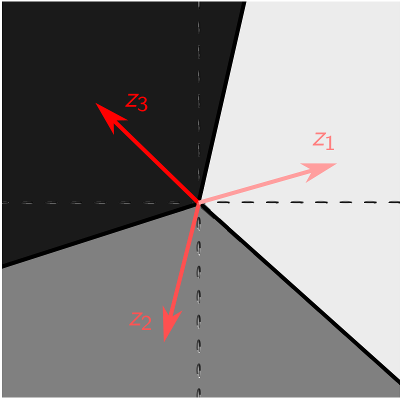

Figure : Propeller Partition in $\mathbb{R}^2$, with moment vectors.

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In $\mathbb{R}^{k-1}$ if we allow $k \geq 3$ pieces, what is the best?

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In $\mathbb{R}^{k-1}$ if we allow $k \geq 3$ pieces, what is the best?
Conjecture [Khot and Naor, '09,'11]: The planar propeller $\times \mathbb{R}^{k-3}$.

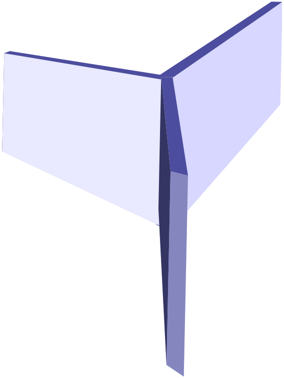

Figure : The Propeller Partition in $\mathbb{R}^3$

Kernel Clustering: An Example

A, $n \times n$ matrix, "correlation data." B, $k \times k$ matrix, "hypothesis."

Partition is variable.

E.g. $n = 9$, $k = 3$. Partition $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ into 3 sets.

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A Second Example

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$$\longmapsto \left(\begin{array}{c|c|c} \sum_{i,j=1}^{2} a_{ij} & \sum_{i=1}^{2}\sum_{j=3}^{6} a_{ij} & \sum_{i=1}^{2}\sum_{j=7}^{9} a_{ij} \\ \hline \sum_{i=3}^{6}\sum_{j=1}^{2} a_{ij} & \sum_{i,j=3}^{6} a_{ij} & \sum_{i=3}^{6}\sum_{j=7}^{9} a_{ij} \\ \hline \sum_{i=7}^{9}\sum_{j=1}^{2} a_{ij} & \sum_{i=7}^{9}\sum_{j=3}^{6} a_{ij} & \sum_{i,j=7}^{9} a_{ij} \end{array}\right)$$

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A Third Example

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\end{array}\right)$$

Form the pairs of the 3 sets.

$\{1, 3\} \times \{1, 3\}$, $\{1, 3\} \times \{2, 4, 5, 6\}$, $\{2, 4, 5, 6\} \times \{1, 3\}$, ...

These product sets partition A.

The Kernel Clustering Problem

Maximize

$$\sum_{i,j=1}^{k} b_{ij} \left(\sum_{(p,q)\in S_i \times S_j} a_{pq} \right)$$

over all partitions $\{S_1, \ldots, S_k\}$ of $\{1, \ldots, n\}$.

$A = \{a_{pq}\}_{p,q=1}^{n}$, symmetric positive semidefinite , $\sum_{p,q=1}^{n} a_{pq} = 0$.
$B = \{b_{ij}\}_{i,j=1}^{k}$ symmetric positive semidefinite.

Introduced by [Song, Smola, Gretton and Borgwardt, '07]

A special case: MAXCUT

Maximize

$$\sum_{i,j=1}^{2} b_{ij} \left(\sum_{(p,q)\in S_i \times S_j} a_{pq} \right) \qquad (*)$$

over all partitions $\{S_1, S_2\}$ of $\{1, \ldots, n\}$.

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Let $G = (V, E)$ be a graph, $V = \{1,\ldots,n\}$, without loops. Let $V = S_1 \cup S_2$ be a partition. Let A be the Laplacian of G, so that

$$a_{ij} = \begin{cases} \mathrm{degree}(i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } (i,j) \in E \\ 0 & \text{if } i \neq j \text{ and } (i,j) \notin E \end{cases}$$

A is positive semidefinite, and $\sum_{i,j=1}^{n} a_{ij} = 0$.

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Let $G = (V, E)$ be a graph, $V = \{1, \ldots, n\}$, without loops. Let $V = S_1 \cup S_2$ be a partition.
For $i \in \{1, \ldots, n\}$, let $\varepsilon_i = 1$ if $i \in S_1$. Let $\varepsilon_i = -1$ if $i \in S_2$. Then

$$\sum_{i,j=1}^{2} b_{ij} \left(\sum_{(p,q)\in S_i\times S_j} a_{pq} \right) = \sum_{i,j=1}^{n} a_{ij}\varepsilon_i\varepsilon_j = \cdots = 4\,|E(S_1, S_2)|$$

So, maximizing $(*)$ computes $4\,\mathrm{MaxCut}(G)$.

A simpler case: B is the identity

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$$\sum_{i,j=1}^{k} b_{ij} \left(\sum_{(p,q)\in S_i \times S_j} a_{pq} \right)$$

over all partitions $\{S_1, \ldots, S_k\}$ of $\{1, \ldots, n\}$.

Let $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, or $B = k \times k$ identity matrix.

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Maximize the intra-cluster correlations:

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This is the simplest nontrivial B and we still don't fully understand it. We now understand the case $k=4$. (The cases $k=2,3$ were treated in [Khot and Naor, '09].)

Geometric Formulation

Theorem (Main theorem; geometric formulation)

Let $\{A_1, \ldots, A_k\}$ *be a partition of* $\mathbb{R}^3$. *Let* $z_i := \int_{A_i} x e^{-(x_1^2+x_2^2+x_3^2)/2} \frac{dx}{(2\pi)^{3/2}} \in \mathbb{R}^3$. *Then*

$$\sum_{i=1}^{k} \|z_i\|_2^2 \leq \frac{9}{8\pi}$$

This bound cannot be improved.

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This bound cannot be improved. To get equality, let $\{P_1, P_2, P_3\}$ be the partition of $\mathbb{R}^2$ into 120° sectors centered at the origin. Let $A_i = P_i \times \mathbb{R}$ for $i \in \{1, 2, 3\}$ and $A_i = \emptyset$ for $i \in \{4, \ldots, k\}$.

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The following were shown to be equivalent to this Theorem in [Khot and Naor, '09,'11]

Complexity Theoretic Formulation

Maximize

$$\sum_{i,j=1}^{4} b_{ij} \left(\sum_{(p,q)\in S_i \times S_j} a_{pq} \right) \qquad (*)$$

over all partitions $\{S_1, \dots, S_4\}$ of $\{1, \dots, n\}$.

Theorem (Main theorem; complexity theoretic formulation)

Let $\mathcal{O}$ be the following optimization problem. The input is an $n \times n$ symmetric positive semidefinite matrix $A = \{a_{pq}\}$ with $\sum_{p,q=1}^{n} a_{pq} = 0$, and also a 4×4 symmetric positive semidefinite matrix $B = \{b_{ij}\}$ with $b_{ii} = 1$ and $\sum_{i,j=1}^{4} b_{ij} = 0$. The goal is to maximize in polynomial time the quantity $(*)$. **Then the UGC hardness threshold of $\mathcal{O}$ equals $\frac{2\pi}{3}$.**

Probabilistic Formulation

Theorem (Main theorem; probabilistic formulation)

Let $(g_1, g_2, g_3, g_4) \in \mathbb{R}^4$ be a mean zero Gaussian vector (with arbitrary covariance matrix). Then

$$\left|\mathbb{E}\left[\max_{i\in\{1,2,3,4\}} g_i\right]\right| \leq \frac{3}{2\sqrt{2\pi}}\sqrt{\sum_{i=1}^{4}\mathbb{E}\left[g_i^2\right]}$$

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This bound cannot be improved.

Analytic Formulation

Theorem (Main theorem; analytic formulation)

Let $\{a_{ij}\}$ *be an* $n \times n$ *positive semidefinite matrix with* $\sum_{i,j=1}^{n} a_{ij} = 0$. *For every* $\{v_1, v_2, v_3, v_4\} \subseteq S^3$ *with* $\sum_{i=1}^{4} v_i = 0$,

$$\max_{x_1,\ldots,x_n \in S^{n-1}} \sum_{i,j=1}^{n} a_{ij} \langle x_i, x_j \rangle \leq \frac{2\pi}{3} \max_{y_1,\ldots,y_n \in \{v_1,v_2,v_3,v_4\}} \sum_{i,j=1}^{n} a_{ij} \langle y_i, y_j \rangle$$

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Usual SDP Rounding

How to relate the complexity theoretic form to the geometric form?

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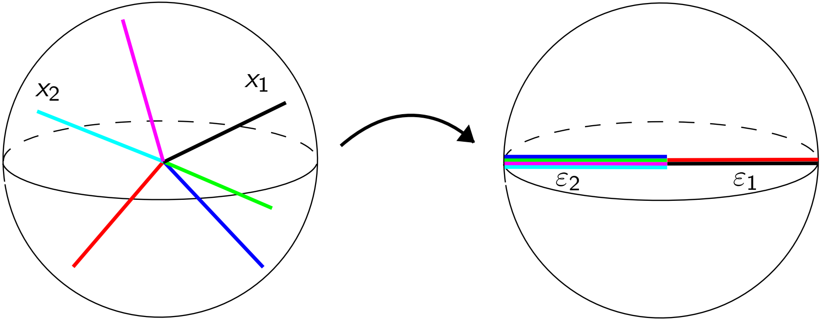

Figure : $\varepsilon_1 = +1$, $\varepsilon_2 = -1$. (Used e.g. in MAXCUT analysis [Goemans and Williamson '95] [Khot, Kindler, Mossel, O'Donnell, '07])

More general SDP rounding: Depiction of Algorithm

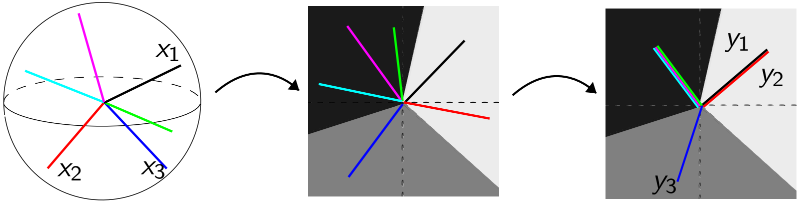

Figure : $y_1 = y_2$ and $y_3 \neq y_2$. (Used in [Khot and Naor, '09, '11])

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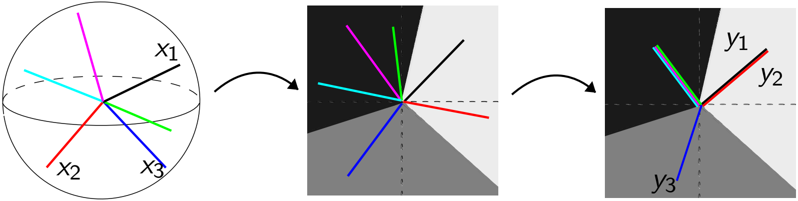

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Let $A_1, \ldots, A_k$ be optimal partition of $\mathbb{R}^{k-1}$. (Maximize the sum of squared Gaussian moments.) Let G be a $(k-1) \times n$ matrix of iid Gaussians. Let $x_i \in S^{n-1}$. Define $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, k\}$ so that $\sigma(i) = p$ where $Gx_i \in A_p$.

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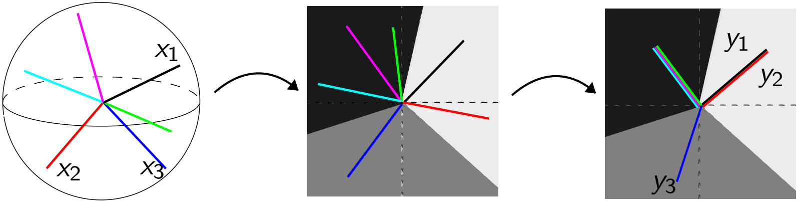

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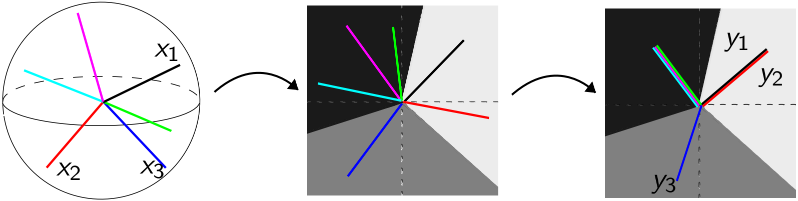
More general SDP rounding: Depiction of Algorithm



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Can bound expected value of algorithm's output using the Propeller Problem.

Reduction to the sphere

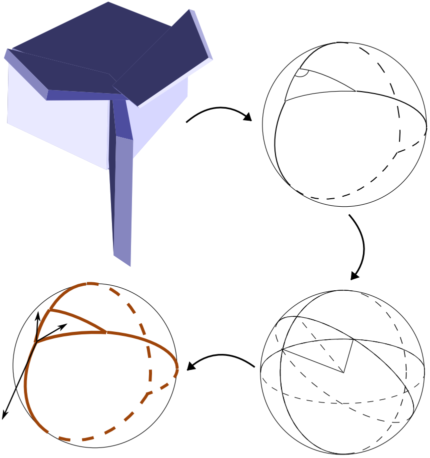

Figure : Analyzing a partition.

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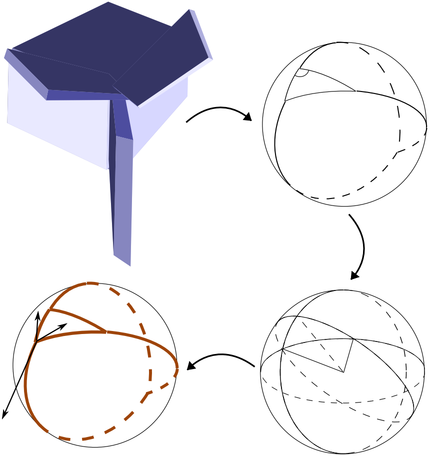


Figure : Analyzing a partition.

Intersect partition of $\mathbb{R}^3$ with the sphere.

Get explicit formula for moment vector $\int_A x dS(x)$.

Replace edges with rubber bands.

Rubber band configurations

An optimal partition is stable under deformation of the rubber bands. That is, if we move their "rest" lengths to be tangent to a vertex, then the sum of these tangent vectors (i.e. the **sum of the forces**) is zero. So an unstable configuration is not optimal.

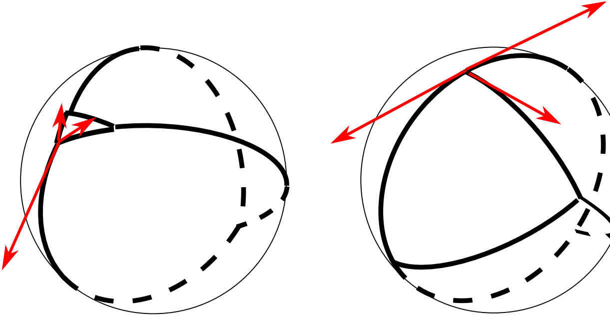


Figure : Unstable rubber band configurations

Numerical Computation

How to analyze the objective function:

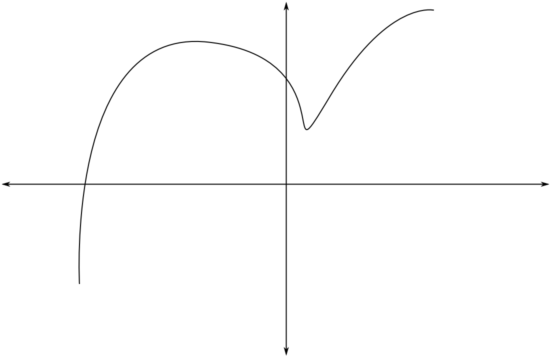
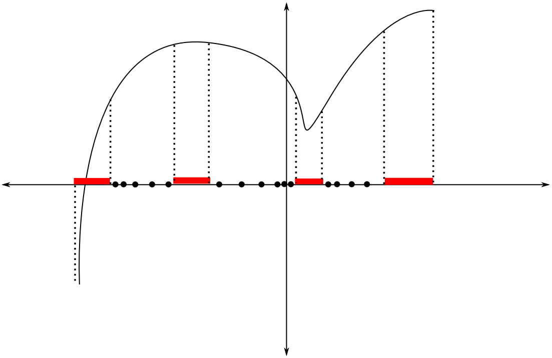


Figure : The sum of the squared moments of the partition of $\mathbb{R}^3$

Numerical Computation

Red regions are treated rigorously, using elementary computations and analogies with rubber bands. The rest uses a brute force epsilon net traversal, with derivative estimates.



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Conclusions

• We better understand the UGC approximation threshold of the following problem: Maximize

$$\sum_{i=1}^{4} \sum_{(p,q)\in S_i\times S_i} a_{pq}$$

over all partitions $\{S_1, \ldots, S_4\}$ of $\{1, \ldots, n\}$.

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• However, it would be nice to have a shorter, non-computer-assisted proof.

• Higher dimensions ($\mathbb{R}^4$ with $k = 5$, ...) are still open.

An Example of Similar Phenomena: K. Ball's Cube Slicing

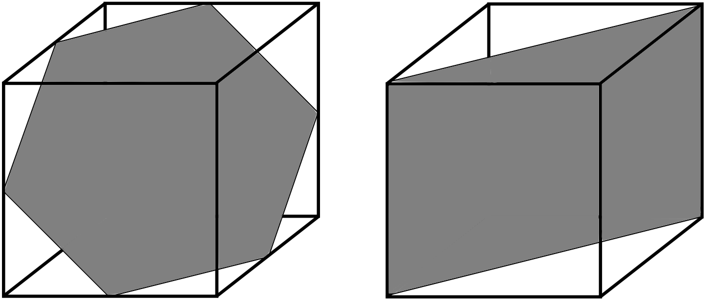

Figure : Two Slices of the Cube. The right one has largest area, for the n-dimensional cube, and hyperplane slices. Solution uses classical Fourier analysis. Reduces problem to a tricky real variable optimization problem.

Review of Khot and Naor, '09,'11

Theorem (Grothendieck Inequality, '53)

There exists $K > 0$ such that, for $a_{ij} \in \mathbb{R}$,

$$\max_{\substack{x_1,\ldots,x_n,y_1,\ldots,y_n \in \ell_2 \\ ||x_i||=||y_i||=1}} \sum_{i,j=1}^{n} a_{ij}\langle x_i, y_j\rangle \leq K \cdot \max_{\varepsilon_1,\ldots,\varepsilon_n,\delta_1,\ldots,\delta_n \in \{\pm 1\}} \sum_{i,j=1}^{n} a_{ij}\varepsilon_i\delta_j$$

Left side is an SDP, right side is an integer program.

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Left side is an SDP, right side is an integer program.
It seemed for a while that projecting onto a random line (after "preprocessing") was the best thing to do. Now we know that this is not the best. ($1.676 \leq K \leq \frac{\pi}{2\log(1+\sqrt{2})} - \eta$, [Braverman, Makarychev, Makarychev, Naor '11])

Review of Khot and Naor, '09,'11

Theorem (Positive Semidefinite Grothendieck Inequality, Rietz, '74)

Let $\{a_{ij}\}_{i,j=1}^{n}$ *be symmetric positive semidefinite. Then*

$$\max_{x_1,\ldots,x_n\in S^{n-1}} \sum_{i,j=1}^{n} a_{ij}\langle x_i, x_j\rangle \leq \frac{\pi}{2}\cdot \max_{\varepsilon_1,\ldots,\varepsilon_n\in\{\pm 1\}} \sum_{i,j=1}^{n} a_{ij}\varepsilon_i\varepsilon_j$$

The constant $\pi/2$ *is the smallest possible.*

Review of Khot and Naor, '09,'11

Theorem (Generalized Positive Semidefinite Grothendieck Inequality, Khot and Naor, '11)

Let $\{a_{ij}\}_{i,j=1}^{n}$ *be real symmetric positive semidefinite. Let* $\nu_1,\ldots,\nu_k\in\mathbb{R}^k$, $2\leq k\leq n$. *Let* $B=\{b_{ij}\}=\{\langle\nu_i,\nu_j\rangle\}$. *Then*

$$\max_{x_1,\ldots,x_n\in S^{n-1}}\sum_{i,j=1}^{n}a_{ij}\langle x_i,x_j\rangle\leq\frac{1}{C(B)}\cdot\max_{\sigma\colon\{1,\ldots,n\}\to\{1,\ldots,k\}}\sum_{i,j=1}^{n}a_{ij}\langle\nu_{\sigma(i)},\nu_{\sigma(j)}\rangle$$

Let $d\gamma_n(x):=\frac{1}{(2\pi)^{n/2}}e^{-(x_1^2+\cdots+x_n^2)/2}dx$. *The sharp constant is*

$$C(B):=\sup_{\substack{(f_1,\ldots,f_k)\colon\\ f_i\in L_2(\gamma_n)\\ f_i\geq 0,\sum f_i=1}}\sum_{i,j=1}^{k}b_{ij}\left\langle\int_{\mathbb{R}^n}xf_i(x)d\gamma_n(x),\int_{\mathbb{R}^n}xf_j(x)d\gamma_n(x)\right\rangle$$

How is the Propeller Problem different from classical isoperimetric problems?

In classical isoperimetry, you can zoom in on singularities. Then you can restrict what kinds of singularities occur.

In this problem, it seems unnatural to do this argument.

Why is the Propeller Problem Difficult?

What usually works:
Symmetrization

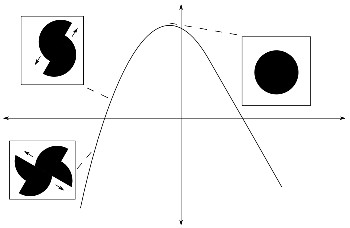

Figure : Symmetrization for Isoperimetry in $\mathbb{R}^2$. Symmetrization works when exactly one local maximum/minimum exists.

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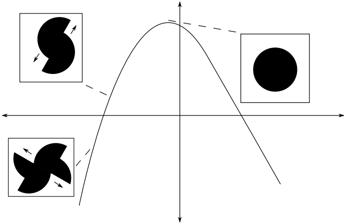

Figure : Symmetrization for Isoperimetry in $\mathbb{R}^2$. Symmetrization works when exactly one local maximum/minimum exists.

However, we have at least 2 local maxima. So symmetrization seems difficult to use directly.

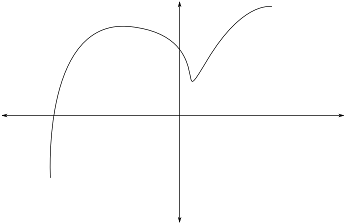

Figure : Objective Function