

THE RADON AND FOURIER TRANSFORMS: THE MATHEMATICS OF X-RAYS AND CT-SCANS

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ABSTRACT. These notes provide an introduction to the mathematics used in medical imaging. In Section 6 we review the basics of calculus and multivariable calculus. In Section 2 we introduce Fourier Series, which is a premonition for the introduction of the Fourier Transform in Section 3. Finally, we will treat the mathematics of CT-Scans with the introduction of the Radon Transform in Section 4. Section 8 describes the notation used throughout, with a Bibliography appearing afterwards. For an excellent and thorough treatment of these topics, see [SS].

Below, there will be a lot of “transforms” discussed below. What exactly is a transform? Well, it is quite simple. You take a function, and then you transform it into another function.

In all, calculus (in its modern form) was discovered over 300 years ago. Fourier series have been around for over 200 years. The Radon Transform was discovered around 100 years ago. Imagine what has happened since then.

1. INTRODUCTION

How exactly do these medical imaging devices work? We attempt to answer this question below. Along the way, we will survey related problems from calculus, signal processing, geometry, etc.

Let us begin with a simple example. Suppose I get an X-ray of my hand. The end result is an image that shows the bones in my hand, and the outlines of some of the soft tissues. How did this imaging process work? Well, X-rays are sent onto your hand with a certain intensity, and then the X-rays that manages to pass through your hand are recorded on an X-ray sensitive sheet.

Let us now examine things more carefully. Suppose a single beam of X-rays passes through a portion of your hand. Suppose the beam has intensity I_0 before entering your hand, and it has intensity I after exiting your hand. Then, if we assume the hand is homogeneous, the following holds

$$(1.1) \quad I = I_0 e^{-d\rho}$$

where d is the distance that the beam travels through your hand, and ρ is the **attenuation** coefficient, which depends on the density of your hand, etc. However, my hand is not homogeneous. The bones and softer tissues interact differently with the X-rays (if not, this imaging process would be useless). Thus, if the beam travels along a line L , we can think of $\rho(x)$ as the attenuation coefficient at the point x

Date: April 17, 2010.

These notes accompany a lecture given for cSplash, a single day enrichment event for high school students. Most of the material is taken from [SS], Chapter 6.5. Steven M. Heilman is a graduate student of mathematics at the Courant Institute of Mathematical Sciences, New York University. His email address is heilman@cims.nyu.edu.

in L . Moreover, we can think of ρ as having a constant value along very short distances of L , and we are led to the formula

$$(1.2) \quad I = I_0 e^{\int_L \rho(x) dx}$$

Now, the X-ray image I described above “flattened” my hand in a sense. That is, we lost some information with our imaging technique. We would really like to see small features of my hand in any small volume. To be more precise, we would like to take some measurements and then exactly reconstruct my hand (in three dimensions). You can draw an analogy here with a typical photograph. A photograph is a “flattened” two-dimensional image. You can’t examine the edges of my fingers from a photograph of my palm. We are now ready to state our main question:

Problem 1.1. Is there a system of X-ray measurements that we can take, for which we can reconstruct my hand exactly?

The answer to this question is remarkably simple (with an appropriate understanding of preliminaries), and it underlies the workings of both CT-scans and MRIs. In order to achieve our affirmative answer, we must first listen to our intuition.

2. FOURIER SERIES

How exactly does the human ear turn sound into that thing that we hear? The outer ear captures the sound and eventually transfers the vibrations to the cochlea. Within the cochlea are thousands of tiny hairs, configured in such a way that the frequencies of the sound can be extracted. Specifically, certain hairs are tuned to specific frequencies (by their location within the cochlea). When a hair detects the frequency to which it is tuned, it tells your brain that you are hearing that particular frequency (at a particular volume).

In mathematical terms, the ear decomposes an audio signal into its **Fourier series**. Let’s make this statement more precise. Suppose $g: [0, 1] \rightarrow \mathbb{R}$ is a function such that $g(0) = g(1) = 0$. Define $f: [-1, 1] \rightarrow \mathbb{R}$ as the odd extension of g by the rule $f(-x) = -g(x)$ for $x \in [0, 1]$. Then, assuming further that f is differentiable¹, we may write

$$(2.1) \quad f(x) = \sum_{k=1,2,\dots} b_k \sin(\pi k x)$$

where the coefficients b_k are to be determined. To find the b_k , multiply both sides by $\sin(\pi j x)$ for j a positive integer, and integrate both sides to get

$$\begin{aligned} \int_{-1}^1 f(x) \sin(\pi j x) dx &= \sum_{k=1}^{\infty} b_k \int_{-1}^1 \sin(\pi j x) \sin(\pi k x) dx \\ &= b_j \int_{-1}^1 (\sin(\pi j x))^2 dx \end{aligned}$$

¹One can make weaker assumptions that differentiability here. The weakest assumptions one can make, and the different interpretations of the equality sign in Eq. 2.1 provide one with very deep questions, which we do not have time to answer.

where the last equation follows since $\int_{-1}^1 \sin(\pi jx) \sin(\pi kx) dx = 0$ for $j \neq k$ (Exercise). (Here we have also assumed that we can interchange the sum and the integral, which requires certain assumptions). Thus, since $\int_{-1}^1 (\sin(\pi jx))^2 dx = 1$, we get

$$(2.2) \quad b_k = \int_{-1}^1 f(x) \sin(\pi kx) dx$$

Combining Eqs. 2.1 and 2.2, we see that we can find the “frequency” components of f exactly. To make the connection with the human ear more explicit, think of a rope (or Slinky) whose ends are fixed. Think of our rope as the interval $[0, 1]$ used above. What happens when we shake the rope up and down? If we do this in a random way, the rope will bounce up and down in a seemingly unpredictable way². However, if we shake the rope up and down at a specific, regular rate, we will get a sine curve. In fact, you can (theoretically) get any sine curve $\sin(\pi kx)$ for any positive integer k . This set of sine curves forms the resonant modes of the rope, or the harmonics.³

Equation 2.1 essentially says that *any* vibration that the rope makes is an infinite sum (or superposition) of the harmonics $\sin(\pi kx)$. Strictly speaking, we have not proven this assertion, but it is true, if properly interpreted. If we turn our observations around, we see that if we know the coefficients (i.e. the intensity of each frequency) of f , then we can recover f exactly, via Eq. 2.1.

Now that we understand the definition of Fourier series, we are ready to broaden out perspective a bit. Suppose we have $f: [-1, 1] \rightarrow \mathbb{R}$ as above. Write f as the sum of an even function and an odd function: $f(x) = (f(x) + f(-x))/2 + (f(x) - f(-x))/2$. Analogous to the above, we may write

$$(2.3) \quad f(x) = \left(\sum_{k=0}^{\infty} a_k \cos(\pi kx) \right) + \left(\sum_{k=1}^{\infty} b_k \sin(\pi kx) \right)$$

under suitable hypotheses on f . Doing an analogous computation to the above allows us to find the coefficients a_k and b_k . Moreover, via the complex exponential (Eq. 6.1), it is a bit more convenient to re-write Eq. 2.3 for $g: [0, 1] \rightarrow \mathbb{R}$ as

$$(2.4) \quad g(x) = \sum_{k=-\infty}^{\infty} \gamma_k e^{2\pi i k x}$$

The convenience here can be understood from the nice formula for the coefficients γ_k

$$(2.5) \quad \gamma_k = \int_0^1 g(x) e^{-2\pi i k x} dx$$

(To get this formula, multiply both sides of Eq. 2.4 by $e^{-2\pi i j x}$, etc.) It turns out that the real (2.3) and complex (2.4) Fourier series are essentially equivalent.

Exercise 2.1. Let $f(x) = x$, $f: [-1, 1] \rightarrow \mathbb{R}$. Compute the coefficients (the Fourier coefficients) of the expansion of f using Eq. 2.3 and Eq. 2.4. Is there a relation between the a_k , b_k and γ_k ?

²To make the connection between Fourier series and traveling waves more explicit requires the analysis of the wave equation in one dimension, which we do not have time to discuss.

³Harmonics are often manipulated in musical instruments. For example, John Coltrane was known for his experiments with the harmonics of the saxophone, demonstrated, e.g., in the aptly named song “Harmonique.”

See section 7 for a discussion on the assumptions needed to make Eq. 2.4 rigorous. Before we move on, we will give a geometric application of Fourier series.

2.1. Isoperimetric Problem. Let's say I have one hundred meters of fence, and I want to build a fence on flat ground which encloses the most area (say, so my cows have the most available grazing space). How should I build my fence? After some thought, you may realize that the fence should be a circle. Can you justify your answer? The solution to this problem was “known” to several ancient cultures, but a rigorous solution was not known until much more recently. Below we will sketch such a rigorous solution to the following problem:

Problem 2.2. Let Γ be a curve in the plane of fixed length 1, which does not intersect itself. What is the maximum area A that γ can enclose?

Now, suppose the curve Γ is the graph of a continuous function $\gamma: [0, 1] \rightarrow \mathbb{R}^2$ where $\gamma(0) = \gamma(1)$. Thus, we can write $\gamma = \gamma(t) = (x(t), y(t))$, and we have $\Gamma = \{(x(t), y(t)): t \in [0, 1]\}$. Assume further that the curve γ has a definable length. That is, given any numbers so that $0 = t_0 \leq \dots \leq t_N = 1$, we have

$$\sum_{i=1}^N |\gamma(t_i) - \gamma(t_{i-1})| < C = C(\gamma)$$

Assume now also that $\gamma(t)$ travels along Γ at unit speed. This amounts⁴ to assuming that $\sqrt{(x'(t))^2 + (y'(t))^2} = 1$, since from the arc length formula, arc length is given by

$$1 = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

and $\sqrt{(x'(t))^2 + (y'(t))^2}$ is the “infinitesimal velocity” at $\gamma(t)$. Actually, since $(x'(t))^2 + (y'(t))^2 = 1$, we also have

$$(2.6) \quad 1 = \int_0^1 (x'(t))^2 + (y'(t))^2 dt$$

Finally before we begin the proof, we need to have a way to measure the area enclosed by Γ . Suppose D is a bounded region in the plane⁵ so that the boundary of D is Γ . We then have

$$\begin{aligned} A &= \left| \int_D dx dy \right| \\ &= \left| \int_D d\left(\frac{1}{2}(x dy - y dx)\right) \right|, \text{ by differential forms arithmetic} \\ &= \left| \int_{\Gamma} \frac{1}{2}(x dy - y dx) \right|, \text{ by “fund. thm. of calculus” (Stokes' Theorem)} \\ &= \frac{1}{2} \left| \int_{[0,1]} (x(t)y'(t) - y'(t)x(t)) dt \right| \end{aligned}$$

To summarize,

$$(2.7) \quad A = \frac{1}{2} \left| \int_{[0,1]} (x(t)y'(t) - y'(t)x(t)) dt \right|$$

⁴Justifying this sentence requires a remarkable amount of work, but it can be done.

⁵See previous footnote.

We are now ready to prove the following

Theorem 2.3. *Suppose Γ is a curve in the plane of length 2π which does not intersect itself. Then $A \leq 1$.*

Proof:

We will see that γ will have maximal area when the Fourier expansions of x and y collect onto the lowest frequencies. This will follow by manipulating the formulas we derived above.

First, suppose $x(t) = \sum_{n \in \mathbb{Z}} a_n e^{int}$ and $y(t) = \sum_{n \in \mathbb{Z}} b_n e^{int}$. By differentiating term by term, we also have $x'(t) \sim \sum_{n \in \mathbb{Z}} in a_n e^{int}$ and $y'(t) \sim \sum_{n \in \mathbb{Z}} in b_n e^{int}$. With these equations, Parseval's Identity, and Eq. 2.6 (for the length 2π case), we get

$$(2.8) \quad 1 = \sum_{n \in \mathbb{Z}} |n|^2 (|a_n|^2 + |b_n|^2)$$

Again apply Parseval's Identity (bilinear form) to Eq. 2.7 (and again use the Fourier expansions of x, y and their derivatives) to get

$$(2.9) \quad A = \pi \left| \sum_{n \in \mathbb{Z}} n(a_n \bar{b}_n - b_n \bar{a}_n) \right|$$

(To see where the complex conjugates come from, observe that $x(s)$ and $y(s)$ are real valued, so $x(s) = \overline{x(s)}$, and similarly for y and the derivatives x' and y' . Thus, we can apply the bilinear form of Parseval's identity after taking the complex conjugates of the Fourier expansions used above, etc. Also, the $\pi = 2\pi/2$ comes directly from Parseval's identity, and because we made x and y defined on the interval $[0, 2\pi]$.) Now, since $|z|^2 = |z\bar{z}|$ for a complex number z , we see (using the triangle inequality) that

$$(2.10) \quad |a_n \bar{b}_n - b_n \bar{a}_n| \leq 2|a_n||b_n| \leq |a_n|^2 + |b_n|^2$$

where the last inequality follows from the arithmetic mean/geometric mean inequality. Thus, using this inequality on Eq. 2.9, the triangle inequality, the easy fact that $|n| \leq |n|^2$, and Eq. 2.8 all give

$$(2.11) \quad A \leq \pi \left(\sum_{n \in \mathbb{Z}} n^2 (|a_n|^2 + |b_n|^2) \right) = \pi$$

Up until this point, we have really just used our assumptions about the problem. We are now poised to optimize the coefficients a_n and b_n so as to maximize A . So, suppose $A = \pi$. Then the inequality of Eq. 2.11 must be an equality. Now, note that, if a_n or b_n is nonzero for some n with $|n| > 1$, then it is impossible to achieve equality (recall we used the fact that $|n| \leq n^2$). Thus, equality can only be achieved if $a_n = b_n = 0$ for all n such that $|n| > 1$. Thus, we must have

$$x(t) = a_{-1}e^{-is} + a_0 + a_1e^{is} \quad \text{and} \quad y(s) = b_{-1}e^{-is} + b_0 + b_1e^{is}$$

From here we can essentially determine these coefficients from formulas that we already derived. Since x, y are real valued, we know that $a_{-1} = \bar{a}_1$ and $b_{-1} = \bar{b}_1$. Now, plug these conditions back into Eqs. 2.8 and 2.10 (and use the fact that we

must have equality) to see that $2(|a_1|^2 + |b_1|^2) = 1$ and $|a_1| = |b_1| = 1/2$. Using Eq. 2.10 again allows us to finally deduce

$$x(t) = a_0 + \cos(\alpha + t) \text{ and } y(t) = b_0 \pm \sin(\alpha + t)$$

i.e. Γ must be a circle.

QED

Exercise 2.4. For a curve of any length, what is the corresponding result?

Note that this proof gives us some nice intuition about the derivative: it amplifies higher frequencies more than lower frequencies. Specifically, frequency n is amplified by n . We will see this effect again in the following sections.

3. FOURIER TRANSFORM

Something from Section 2 may be a bit odd. If you think of a sound wave, there should be infinitely many possible frequencies which can be attained (not just the integer frequencies). And not every function is defined just on an interval (consider for example $f(t) = 1/(1+t^2)$). Thus, we should really have a good way of generalizing Fourier series to functions on the real line. So, let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a real valued function⁶. We can make such a generalization of Eqs. 2.4 and 2.5 by replacing the sums with integrals:

$$(3.1) \quad \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \xi} dx$$

Thus, we interpret $\hat{f}(\xi)$ (the **Fourier Transform** of f) as the frequency component of f with frequency ξ . Under suitable assumptions, we can recover f from its frequencies, just as before:

$$(3.2) \quad f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x \xi} d\xi$$

As we observed above, taking a derivative of f accentuates the higher frequencies of f , and we observe the same thing here, by differentiating Eq. 3.1 under the integral sign

$$(3.3) \quad \frac{d}{dx} \widehat{f}(\xi) = (-2\pi i x) \widehat{f'(x)}$$

and similarly, by integrating $\int f'(x)e^{-2\pi i x \xi}$ by parts, we get

$$(3.4) \quad \widehat{f'(x)} = 2\pi i \xi \hat{f}(\xi)$$

(Compare this to taking a derivative of Eq. 2.4). To build our intuition a bit, let us consider a few simple examples.

⁶For several reasons it is convenient to assume that f is differentiable infinitely many times, and it decays exponentially fast. In other words, we will (implicitly) assume that f is in the Schwartz class, which we denote by $f \in \mathcal{S}(\mathbb{R})$. Actually, this assumption is made on all functions within these notes, unless stated otherwise.

Example 3.1. Let $f(t)$ denote the number of observed sunspots of the Sun, where t is the year in the Gregorian calendar (so if it is April 1st 2010, t should be about 2010.25). As is well known, the number of sunspots rise and fall on an 11 year cycle. That is, the temporal distance between one “solar maximum” and the next is eleven years. Thus, $\hat{f}(\xi)$ will have a large spike at $\xi = 1/11$ (recall that the frequency of a wave is the inverse of the period).

Example 3.2. Let $f(t) = 1$ on $[-1/2, 1/2]$ and equal to zero otherwise. One can find that $\hat{f}(\xi) = \frac{\sin(\pi\xi)}{\pi\xi}$. What can we deduce about the behavior of \hat{f} ? Well, it decays rather slowly. Also, while f is zero outside of an interval, \hat{f} is nonzero essentially everywhere. This fact is exactly related to uncertainty principles. One cannot have both f and \hat{f} localized in a small region. There are in fact many uncertainty principles which exist (In fact, the way that one deduces 3.2 from 3.1 is essentially the manipulation of an uncertainty principle.) Perhaps the best known uncertainty principle is the following:

Theorem 3.3. (*Heisenberg Uncertainty Principle*) Suppose ψ satisfies $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$. Then the following holds

$$\left(\int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx \right) \left(\int_{-\infty}^{\infty} \xi^2 |\hat{\psi}(\xi)|^2 d\xi \right) \geq \frac{1}{16\pi^2}$$

and equality only holds if $\psi = Ae^{-Bx^2}$ where $B > 0$ and $|A|^2 = \sqrt{2B/\pi}$

One proves this by integrating by parts ($1 = \int_{\mathbb{R}} |\psi(x)|^2 dx = -\int_{\mathbb{R}} x \frac{d}{dx} |\psi(x)|^2 dx$, etc.), applying Cauchy Schwarz, and Eq. 3.3. (Here we have assumed that ψ decays rapidly. See the footnote above.) Note that one can find the more general result by applying the above inequality to $e^{-2\pi i x \xi_0} \psi(x + x_0)$ and changing variables to get

$$\left(\int_{-\infty}^{\infty} (x - x_0)^2 |\psi(x)|^2 dx \right) \left(\int_{-\infty}^{\infty} (\xi - \xi_0)^2 |\hat{\psi}(\xi)|^2 d\xi \right) \geq \frac{1}{16\pi^2}$$

Example 3.4. Think of $f(t)$ as a sound signal. On CDs, only discrete samples of a sound wave are stored. How is it possible to recover the original sound wave from just a *finite* number of samples? In general this is impossible. But think about what sounds you can hear. You can’t actually hear all frequencies of sound, only those between 20 and 20,000 Hz. So, in attempting to reconstruct f from a finite number of samples, we may assume that $\hat{f} = 0$ on frequencies outside this range. If we make this assumption, then we can actually recover f , as in the following statement:

Theorem 3.5. (*Shannon Sampling Theorem*) Let $f(t)$ be such that $\hat{f} = 0$ outside of the interval $[-1/2, 1/2]$. Then

$$f(x) = \sum_{n \in \mathbb{Z}} f(n) K(x - n)$$

where $K(y) = \frac{\sin \pi y}{\pi y}$.

To prove this, one only needs to apply the Poisson summation formula and Eq. 3.1 (the Fourier inversion formula). (The Poisson summation formula we need is: $\hat{f}(\xi) = \sum_{n \in \mathbb{Z}} f(n) e^{-2\pi i n \xi}$) (We also need the following facts: $g(x) \widehat{e^{-2\pi i x h}} = \hat{g}(\xi + h)$, which follows from the definition of the Fourier transform.)

Before we begin our study of the Radon transform, we will briefly mention how the Fourier transform is defined in higher dimensions. Let d be a positive integer. Then the Fourier transform is defined as follows (compare to Eqs. 3.1,3.2)

$$(3.5) \quad \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i(x \cdot \xi)} dx$$

and the Fourier inversion formula holds (under appropriate assumptions)

$$(3.6) \quad f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)e^{2\pi i(x \cdot \xi)} d\xi$$

4. RADON TRANSFORM

We are finally ready to analyze Problem 1.1. Can we reconstruct my hand by a series of X-ray measurements? The answer is yes, but how do we go about doing this? Well, first let us convince ourselves that we have enough data to do the reconstruction procedure. Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be the density of my hand. That is, suppose $f(x, y, z)$ is the density of my hand at the spatial position (x, y, z) . Also, assume that we have sent a beam of X-rays along every line in \mathbb{R}^3 . From our discussion above, we saw that this is akin to knowing the integral of f along each line. To expect our reconstruction procedure to be valid, we should count at least three parameters which describe this set of lines. In fact, there are four parameters (say, the point where the line intersects the x-y plane, and then the direction of the line), so there is more information than we need. Therefore, let us simplify the available information slightly.

Suppose we only know the integral of f along each plane. Let $t \in \mathbb{R}$, $\gamma \in S^2$, and let $P_{t,\gamma} = \{x \in \mathbb{R}^3: x \cdot \gamma = t\}$ be the plane at distance t from the origin, with perpendicular in direction γ . (Thus, $P_{t,\gamma} = P_{-t,-\gamma}$). We define the **Radon transform** as follows

$$(4.1) \quad \mathcal{R}(f)(t, \gamma) = \int_{P_{t,\gamma}} f$$

(here $\int_{P_{t,\gamma}} f = \int_{\mathbb{R}^2} f(t\gamma + u_1e_1 + u_2e_2) du_1 du_2$). By explicit calculation, we find that

$$(4.2) \quad \hat{\mathcal{R}}(f)(s, \gamma) = \hat{f}(s\gamma)$$

where the left hand side is the one-dimensional Fourier transform (with respect to s) and the right hand side is the three-dimensional Fourier transform. (This calculation involves writing out the definitions, and applying a change of variables.)

Given a function $F: \mathbb{R} \times S^2 \rightarrow \mathbb{R}$ we define its **dual Radon transform** as follows

$$(4.3) \quad \mathcal{R}^*(F)(x) = \int_{S^2} F(x \cdot \gamma, \gamma) d\sigma(\gamma)$$

(Note that $\langle \mathcal{R}f, F \rangle_{L^2(\mathbb{R} \times S^2)} = \langle f, \mathcal{R}^*F \rangle_{L^2(\mathbb{R}^3)}$. That is, the duality between \mathcal{R} and \mathcal{R}^* comes from interchanging a few integrals. This follows fairly quickly from the definitions of the respective inner products. Also, $d\sigma$ is the area element on the two-dimensional sphere, S^2 . In other words, it is the thing we integrate against.)

Let $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ be the Laplacian on \mathbb{R}^3 (a sum of second derivatives). The following is the reconstruction procedure that we are after. Recall that f represents the density of an object, and $\mathcal{R}(f)$ is the data that we are given. Thus, we need to do something to $\mathcal{R}(f)$ and get f back.

Theorem 4.1. *If f is suitably differentiable, then*

$$\Delta(\mathcal{R}^*\mathcal{R}(f)) = -8\pi^2 f$$

We sketch the proof of this fact. First, apply (one-dimensional) Fourier inversion (i.e. Eq. 3.2) to $\hat{\mathcal{R}}(f)(s, \gamma)$. Thus, from Eq. 4.2, we get that

$$(4.4) \quad \mathcal{R}(f)(t, \gamma) = \int_{-\infty}^{\infty} \hat{f}(s\gamma) e^{2\pi i t s} ds$$

Then, apply \mathcal{R}^* to the both sides of Eq. 4.4 (i.e. apply Eq. 4.3 to the right side of Eq. 4.4). This gives

$$(4.5) \quad \mathcal{R}^*\mathcal{R}(f)(x) = \int_{S^2} \int_{-\infty}^{\infty} \hat{f}(s\gamma) e^{2\pi i(x \cdot \gamma)s} ds d\sigma(\gamma)$$

Then, apply Δ to both sides. Bringing Δ within the integral (we are allowed to carry derivatives within integrals, if we make the correct assumptions) gives

$$\Delta(\mathcal{R}^*\mathcal{R}(f))(x) = \int_{S^2} \int_{-\infty}^{\infty} \hat{f}(s\gamma) (-4\pi^2 s^2) e^{2\pi i(x \cdot \gamma)s} ds d\sigma(\gamma)$$

And then some symbolic manipulation and change of variables gives our desired result.

Heuristically, the transforms \mathcal{R} and \mathcal{R}^* are integration procedures, so they have the opposite effect on f as does the derivative. That is, where we saw from Eq. 3.3 that the derivative accentuates higher frequencies, the operations of \mathcal{R} and \mathcal{R}^* de-accentuate the higher frequencies. Therefore, Theorem 4.1 shows us that the procedure of de-accentuation and accentuation carried out by $\Delta\mathcal{R}^*\mathcal{R}$ exactly cancel each other out. So that is it! Problem solved!

5. CONCLUSIONS

Well okay, we solved our problem, but this is really only the beginning. Fourier analysis has too many applications to mention. (If I could have included a few more nice mathematical applications in these notes, they would have been: solutions of partial differential equations, analytic number theory and theoretical computer science.) But let us not be deceived by the clever solutions and completed problems that we see. There are still advances made, and there are things not understood. In the next chapter, we must hear what we have not yet heard.

6. APPENDIX A: A BRIEF REVIEW OF CALCULUS

Limits, Derivatives, and Integrals

(For more details see any standard introductory calculus textbook, for example, [T].)

Suppose $\{x_1, x_2, \dots\}$ is a sequence of real numbers. We say that x_i approaches the **limit** x if x_i can be made arbitrarily close to x . Formally, this means: for all $\epsilon > 0$, there exists $N_0 = N_0(\epsilon) > 0$ such that $N > N_0$ implies that $|x_N - x| < \epsilon$. We then write $x = \lim_{i \rightarrow \infty} x_i$.

Suppose $f = f(x)$ is a real valued function. That is, for each real number x_0 in some interval (a, b) , $f(x_0)$ is also a real number. Symbolically, we write $f: (a, b) \rightarrow \mathbb{R}$. We say that f has a **limit** L at x_0 if f becomes arbitrarily close to L as we approach x_0 . Formally, we have: for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that: $0 < |x_0 - x| < \delta$ implies $|f(x) - L| < \epsilon$. We then write $\lim_{x \rightarrow x_0} f(x) = L$.

Let $f = f(x)$ be a real valued function. We say that f is **continuous** at x if both sides of the following equation exist, and they are equal.

$$\lim_{x \rightarrow x_0} f(x) = f(x)$$

We say that $f(x)$ has **derivative** at x if the tangent line to the graph of f exists at x . We denote the slope of this line by $f'(x)$ and $\frac{d}{dx}f(x)$. Formally, we define the derivative at x_0 of f as the following limit (if it exists)

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a real valued function such that $f'(x)$ always exists, we say that f is differentiable.

Exercise 6.1. Find a function that is not continuous at a single point. Find a continuous function which is not differentiable at a single point. In all cases, verify your assertions with the formal definitions. Is there a function which is not continuous at any points? Is there a continuous function whose derivative never exists?

Suppose $f: [0, 1] \rightarrow \mathbb{R}$ is a positive, continuous function and $g: [0, 1] \rightarrow \mathbb{R}$ is a positive step function. (We say that g is a step function if looks like a collection of uneven steps. Formally, g is a step function if it can be written as $g(x) = \sum_{i=1}^n s_i(x)$ where $s_i(x) = \lambda_i > 0$ for x in some interval $[x_{i-1}, x_i]$ within $[0, 1]$, and s_i is zero otherwise. We further require that $x_i > x_{i-1}$ and that $\lambda_i = f(x_i^*)$ for some $x_i^* \in [x_{i-1}, x_i]$.) Here we have $1 = x_0 < x_1 < \dots < x_n = 1$. Consider the quantity

$$\sum_{i=1}^n (x_i - x_{i-1})\lambda_i$$

This is the area under g (draw the picture). The **integral** of f on $[0, 1]$ is the area under f . Since g approximates f (by design), we use the area under g to approximate the area under f . Formally, we say the integral $\int_0^1 f(x)dx$ exists if the following limit exists

$$\lim_{(\max_{i=1, \dots, n} |x_i - x_{i-1}|) \rightarrow 0} \sum_{i=1}^n (x_i - x_{i-1})\lambda_i$$

Exercise 6.2. Find a function whose integral does not exist. (Hint: consider a function which is equal to 1 on rational numbers.) How should we define the integral of a function which is not necessarily positive? How can we define the integral of a function which has complex values? (Consider $f(t) = e^{it} = \cos(t) + i \sin(t)$)

Integration by parts is a very useful integration formula. It is essentially the product rule written backwards, but it finds many applications and generalizations. It essentially says that you can move a derivative around within an integral (if you change signs, and incorporate some “boundary terms”). For differentiable functions u, v we have:

$$\int_a^b u(x)v'(x)dx = - \int_a^b u'(x)v(x)dx + (u(b)v(b) - u(a)v(a))$$

Change of variables is another useful formula which also has many generalizations. It allows you to re-arrange the region over which you are integrating.

Suppose $f: [a, b] \rightarrow \mathbb{R}$ and $g: [A, B] \rightarrow [a, b]$ both have continuous derivatives. Also, assume that g is strictly increasing. Then we have:

$$\int_a^b f(x)dx = \int_A^B f(g(y))g'(y)dy$$

i.e.

$$\int_{g(A)}^{g(B)} f(x)dx = \int_A^B f(g(y))g'(y)dy$$

Derivatives and integrals for real-valued functions of more than one real variable use the exact same ideas as in the single variable case. For example, if we have $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, e.g. $f(x, y) = x^2 + xy$, then the derivative in the x direction (denoted by $\frac{\partial}{\partial x}$) is computed in the same way as the single variable derivative with respect to x (if we pretend that y is fixed). Thus, $\frac{\partial}{\partial x}f(x, y) = 2x + y$. To define an integral (say on \mathbb{R}^2), just approximate a given function $f(x, y)$ by a step function as above (where the step function is now constant on rectangles), and then take an appropriate limit. Actually, if you know how to do volumes by slicing or volumes by cylindrical shells (from single variable calculus), you already essentially know how to do multiple integrals (i.e. higher dimensional integrals).

Since the notation is a bit more involved in higher dimensions, we refrain from making precise definitions here. I only want to quickly mention one formula which is used a lot above. It is the higher dimensional version of the change of variables formula. As stated, it will simultaneously introduce some new terminology which may make it look confusing, but the idea is very similar to the one dimensional case. We have a function ϕ which is, say, moving our integration region around in some way. To account for the “distortion” of the volume which results from ϕ , we only need to look at the derivatives of ϕ (which appear on the right hand side of the equality below). With this in mind, I hope that the analogy with the one-dimensional version becomes clearer:

(Change of variables) Let N be a positive integer. Let ϕ be a bijective, C^1 function from an open subset U of \mathbb{R}^N onto an open subset $\phi(U)$ of \mathbb{R}^N . (C^1 means that all first derivatives of ϕ are continuous functions). Assume that the determinant of the Jacobian $\det \text{Jac}(\phi)(\vec{x})$ is nowhere zero. Then

$$\int_{\phi(U)} f(\vec{y})d\vec{y} = \int_U f(\phi(\vec{x})) |\det \text{Jac}(\phi(\vec{x}))| d\vec{x}$$

for $f \geq 0$. (Here $\text{Jac}(\phi)$ is the N by N matrix of partial derivatives of ϕ , and \det is the determinant, which, in a sense, is the Euclidean “infinitesimal volume” element.)

Complex Exponential

Let $i = \sqrt{-1}$. Then the following formula holds

$$(6.1) \quad e^{it} = \cos(t) + i \sin(t)$$

One can justify that both sides are equal by the use of Taylor expansions at zero (Exercise).

7. APPENDIX B: CONVERGENCE OF FOURIER SERIES

Suppose $f: [-\pi, \pi] \rightarrow \mathbb{R}$ is a continuous function such that $f(-\pi) = f(\pi)$, and such that $\int_0^1 |f(x)| dx < \infty$. We want to find conditions on f so that the following holds:

$$(7.1) \quad f(x) = \lim_{N \rightarrow \infty} \sum_{k=-N}^{k=N} \gamma_k e^{ikx} = \sum_{k=-\infty}^{+\infty} \gamma_k e^{ikx}$$

There are several such conditions one can impose on f , and there are several methods to prove that Eq. 7.1 holds in each case. In fact, it took some very sophisticated work to come up with appropriately general assumptions on f to get Eq. 7.1.

Here, in addition to the above assumptions, we will also assume that $\sum_{k=-\infty}^{\infty} |\gamma_k| < \infty$. We will show below that if f has two continuous derivatives, then $\sum_{k=-\infty}^{\infty} |\gamma_k| < \infty$. Thus, as a consequence of what we do below, we will prove the following

Corollary 7.1. *If f has two continuous derivatives (i.e. $f'(x)$ is continuous and differentiable, and $f''(x)$ is continuous), then the Fourier series of f converges absolutely and uniformly to f .*

To achieve our result, we will proceed in two steps. We will first show that it is impossible for the Fourier series to “miss” any of the features of f . If we combine this result with our assumptions above (and some facts about infinite series), we can then get Corollary 7.1. Thus, let us carry out the first step.

Theorem 7.2. *Suppose $f: [-\pi, \pi] \rightarrow \mathbb{R}$ is a continuous function such that $f(-\pi) = f(\pi)$. Assume that $\gamma_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = 0$ for all $n \in \mathbb{Z}$. Then $f = 0$.*

Proof:

We argue by contradiction. Assume without loss of generality that $f(0) > 0$. Since $\gamma_n = 0$ for all $n \in \mathbb{Z}$, any function of the form $p_k(x) = \sum_{j=-k}^{j=k} \alpha_k e^{ijx}$ will satisfy $\int_{-\pi}^{\pi} p_k(x) f(x) dx = 0$. To achieve our contradiction, we will find p_k that “cluster around zero” in such a way that $\int_{-\pi}^{\pi} p_k(x) f(x) dx \rightarrow \infty$ as $k \rightarrow \infty$.

Since f is continuous, we can find a small interval $(-\delta, \delta)$ so that $x \in (-\delta, \delta)$ implies $f(x) > f(0)/2$. Now, let $p(x) = \epsilon + \cos(x)$, where $\epsilon > 0$ is chosen so that $|p(x)| < 1 - \epsilon/2$ (to convince yourself that this is possible, consider the graph of the cosine function). Finally, since $p(0) = 1 + \epsilon$ and p is continuous, we can find another small interval $(-\eta, \eta)$ such that $p(x) \geq 1 + \epsilon/2$ when $x \in (-\eta, \eta)$. (So we must have $0 < \eta < \delta$). Now, define p_k as follows

$$p_k(x) = (p(x))^k$$

By our choice of constants, and by the properties of exponentiation, p_k will grow rapidly around zero, and go to zero rapidly away from zero, which is exactly what we want. Now, recall from above that we have $\int_{-\pi}^{\pi} p_k(x) f(x) dx = 0$ for all k . However, we can split this integral into three regions:

$$\int_{-\pi}^{\pi} p_k(x) f(x) dx = \int_{|x| < \eta} f p_k + \int_{\eta \leq |x| < \delta} f p_k + \int_{\delta \leq |x|} f p_k$$

In the first region, f is bounded from below and so is p_k , giving

$$\int_{|x| < \eta} f p_k \geq 2\eta \frac{f(0)}{2} (1 + \epsilon/2)^k$$

In the second region, f and p are positive, so we know that

$$\int_{\eta \leq |x| < \delta} fp_k \geq 0$$

And in the third region, p_k decays rapidly in k , and f is bounded (it is continuous on a closed interval and hence bounded) so

$$\left| \int_{\delta \leq |x|} fp_k \right| \leq 2\pi \left(\max_{x \in [-\pi, \pi]} |f(x)| \right) (1 - \epsilon/2)^k$$

So if we put the three regions together we see that

$$\int_{-\pi}^{\pi} p_k(x)f(x)dx \geq C(1 + \epsilon/2)^k \rightarrow \infty$$

for some $C > 0$, as $k \rightarrow \infty$. This gives our desired contradiction, and thus proves the theorem.

QED.

We are now ready to prove our convergence result. Let $S_N(f) = \sum_{k=-N}^{k=N} \gamma_k e^{ikx}$ be the N -th partial sum of the Fourier series of f . Then we have the following

Theorem 7.3. *Suppose $f: [-\pi, \pi] \rightarrow \mathbb{R}$ is a continuous function such that $f(-\pi) = f(\pi)$ and such that $\sum_{k=-\infty}^{\infty} |\gamma_k| < \infty$. Then*

$$\lim_{N \rightarrow \infty} S_N(f)(x) = f(x) \quad \text{uniformly in } x$$

Proof:

For any $\epsilon > 0$, we can find N_0 so that $N > N_0$ implies that $\sum_{|k| > N} |\gamma_k| < \epsilon$ (by assumption on the γ_k). Thus, for any $\epsilon > 0$ and any $x \in [-\pi, \pi]$, we can find N_0 so that $N > N_0$ implies that

$$\left| \sum_{|k| > N} \gamma_k e^{ikx} \right| \leq \sum_{|k| > N} |\gamma_k| < \epsilon$$

by the triangle inequality. Note that N_0 does not depend on x . We just showed that $S_N(f)(x)$ converges absolutely for each fixed x , and also that $S_N(f)$ converges uniformly (as a function) to something. How do we find what $S_N(f)$ converges to? Well, each S_N is a continuous function (since it is the sum of a finite number of continuous functions). And we know that the uniform limit of a sequence of continuous functions is continuous (Exercise). Thus, the function $g(x)$ defined by

$$g(x) = \lim_{N \rightarrow \infty} \sum_{k=-N}^{k=N} \gamma_k e^{ikx} = \sum_{-\infty}^{\infty} \gamma_k e^{ikx}$$

exists and is continuous. Also, the k -th Fourier coefficient of g is exactly γ_k . To see this, just use the definition of the Fourier coefficient, and interchange the integral with the infinite sum (which is justified as a consequence of the uniform convergence of the series). Finally, apply Theorem 7.2 to $f - g$. This implies that $f - g = 0$, i.e. $f = g$, as desired.

QED

To Prove Corollary 7.1, it suffices to prove: If f has two continuous derivatives (i.e. $f'(x)$ is continuous and differentiable, and $f''(x)$ is continuous), then the Fourier coefficients of f satisfy $\sum_{k=-\infty}^{\infty} |\gamma_k| < \infty$. But this immediately follows by taking the definition of the γ_k (Eq. 2.5) and integrating by parts twice. (And recalling that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$). Thus, we have proven Corollary 7.1.

8. NOTATION

- (1) $x \in A$ means that x is an element of the set A
- (2) \mathbb{Z} denotes the integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$
- (3) \mathbb{R} denotes the set of real numbers. So, $(-\pi + .3) \in \mathbb{R}$, $7 \in \mathbb{R}$, $e \in \mathbb{R}$, $10^7 + .0001 \in \mathbb{R}$, etc.
- (4) \mathbb{R}^2 denotes the infinite (Euclidean) plane, i.e. $\mathbb{R}^2 = \{(x, y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$
- (5) \mathbb{R}^3 denotes infinite Euclidean space, i.e. $\mathbb{R}^3 = \{(x, y, z) : x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}$
- (6) $f: \mathbb{R}^n \rightarrow \mathbb{R}$ (for, say, $n = 2$) means a function of two variables which takes values in \mathbb{R} . For example, consider $f(x, y) = x^2 + y$ or $g(x, y) = e^{xy} - 2x$.
- (7) $\frac{d}{dx}f(x)$ denotes the first derivative of a function $f: \mathbb{R} \rightarrow \mathbb{R}$
- (8) $\frac{\partial}{\partial x}f(x, y)$ denotes the first derivative in the x -direction of a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$
- (9) \hat{f} denotes the Fourier transform of f
- (10) $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is the two-dimensional sphere
- (11) \mathcal{R} denotes the Radon transform
- (12) \mathcal{R}^* is the dual Radon transform
- (13) $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplacian on \mathbb{R}^n
- (14) $\mathcal{S}(\mathbb{R}^n)$ is the class of real valued functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ which are infinitely differentiable and which decay exponentially fast.

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